THEORY OF PLASTIC FLOW II

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- Review of J_2 theory
- Hugoniot problem
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Review of J_2-theory

Von Mises stress

$$\sigma_e = \sqrt{\tfrac{3}{2} S_{ij} S_{ij}} \quad \Rightarrow \quad F(\sigma_{ij}) = \sigma_e - \sigma_Y \leq 0$$

Review of J_2-theory

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Associated flow rule

$$\dot{\epsilon}_{ij}^{p} = \lambda \frac{\partial F}{\partial \sigma_{ij}} = \frac{3}{2} \frac{\lambda}{\sigma_Y} S_{ij} \qquad \text{(deviatoric)}$$

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Plastic multiplier

$$\lambda = \frac{S_{ij} \dot{\epsilon}_{ij}}{\sigma_Y} \quad \text{(from consistency condition } \dot{F} = 0\text{)}$$

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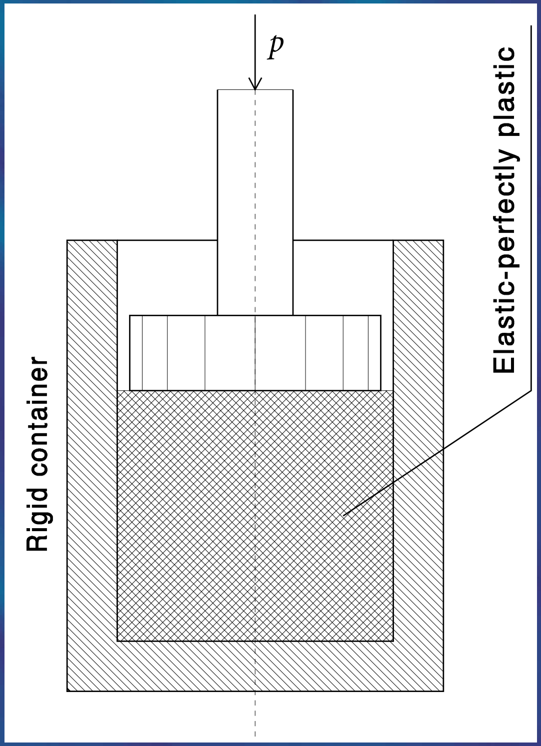
Stress update

$$\dot{\sigma}_{ij} = C_{ijkl}(\dot{\epsilon}_{kl} - \dot{\epsilon}^p_{kl}) = \dot{\sigma}^t_{ij} - 2G\dot{\epsilon}^p_{ij}$$

Hugoniot problem (1/3)



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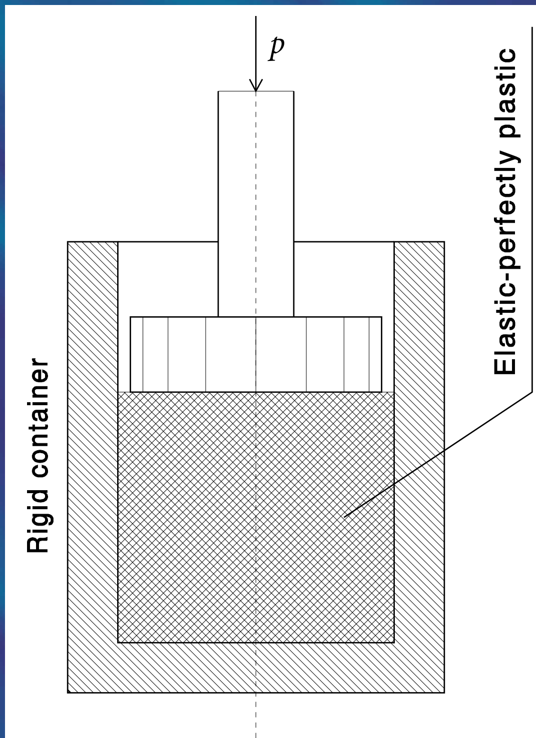

Stress tensor

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_o & 0 & 0 \\ 0 & \sigma_r & 0 \\ 0 & 0 & \sigma_r \end{bmatrix} \qquad (\mu_\sigma = \pm 1)$$

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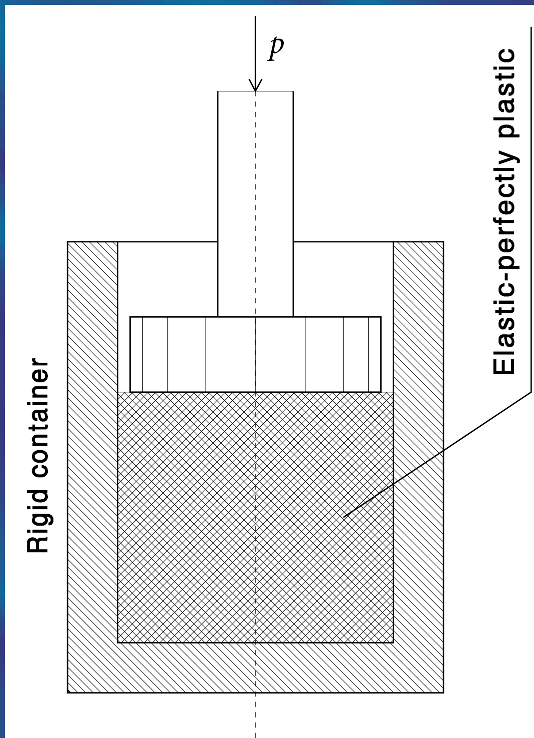

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$$\sigma_e = \tau_e = |\sigma_o - \sigma_r| = \sigma_Y$$

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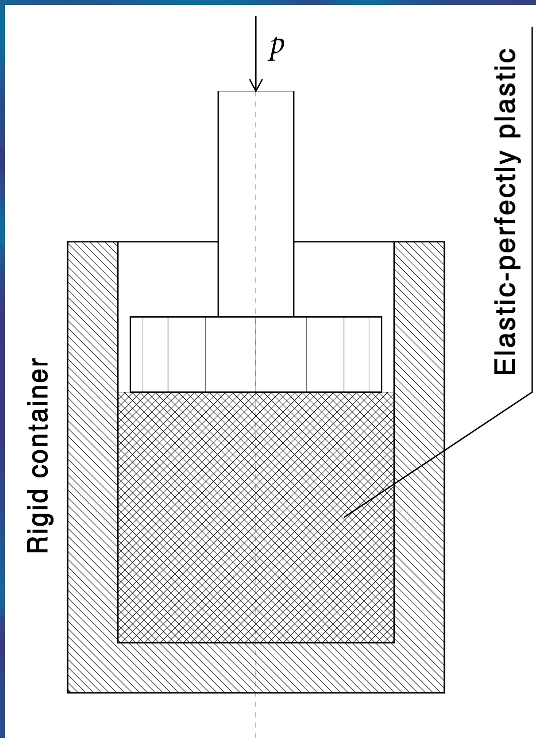

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Boundary condition for elastic state

$$E\epsilon_r^e = \sigma_r - \nu(\sigma_o + \sigma_r) = 0$$

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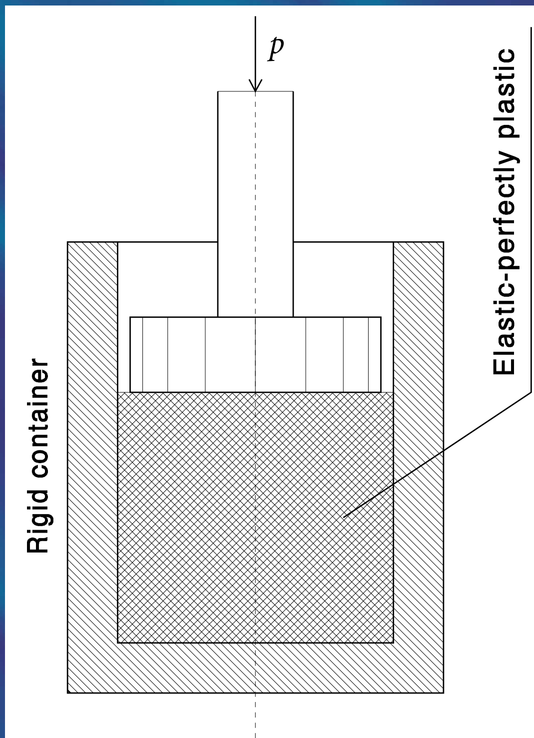

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Hugoniot elastic limit (HEL)

$$p = |\sigma_o| = \frac{1-\nu}{1-2\nu}\sigma_Y$$

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Stress deviator

$$\mathbf{S} = \begin{bmatrix} S_o & 0 & 0 \\ 0 & S_r & 0 \\ 0 & 0 & S_r \end{bmatrix}$$

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Zero trace theorem

$$S_o + S_r + S_r = 0$$

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Stress deviator

$$\mathbf{S} = \frac{S_o}{2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

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Yield condition

$$|S_o - S_r| = \frac{|S_o|}{2}(2+1) = \sigma_Y$$

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therefore

$$\boxed{S_o = -\frac{2}{3}\sigma_Y}$$

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therefore

$$\boxed{S_o = -\frac{2}{3}\sigma_Y}$$

and

$$\boxed{\mathbf{S} = -\frac{\sigma_Y}{3}\mathbf{M}}$$

Remark: Note that the deviator tensor is constant at plastic state.

Hugoniot problem (2/3)

Prandtl-Reuss

$$\dot{\boldsymbol{\epsilon}}^p = \frac{3}{2}\frac{\lambda}{\sigma_Y}\mathbf{S} = -\frac{\lambda}{2}\mathbf{M}$$

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$$\lambda = \frac{S_{ij}\dot{\epsilon}_{ij}}{\sigma_Y}$$

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Plastic strain rate

$$\dot{\boldsymbol{\epsilon}}^p = \frac{\dot{\epsilon}_o}{3}\mathbf{M}$$

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$$\boxed{\Delta\boldsymbol{\epsilon}^p = \frac{\Delta\epsilon_o}{3}\mathbf{M}}$$

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Axial strain

$$\Delta\epsilon_o^e = \Delta\epsilon_o - \Delta\epsilon_o^p = \frac{1}{3}\Delta\epsilon_o$$

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Radial strain

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Elastic strain tensor

$$\Delta\boldsymbol{\epsilon}^e = \frac{1}{3}\Delta\epsilon_o\,\mathbf{I} \qquad \text{(isotropic)}$$

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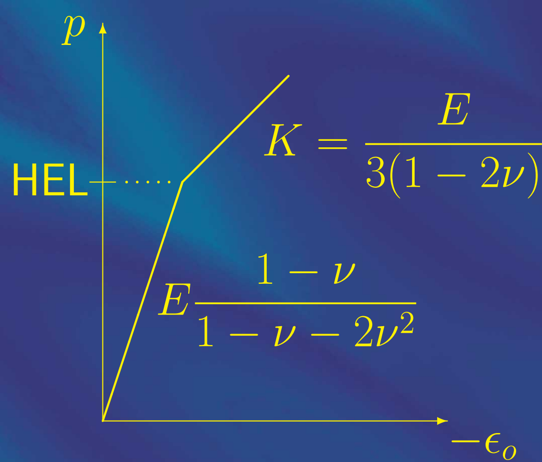
Hydrostatic pressure

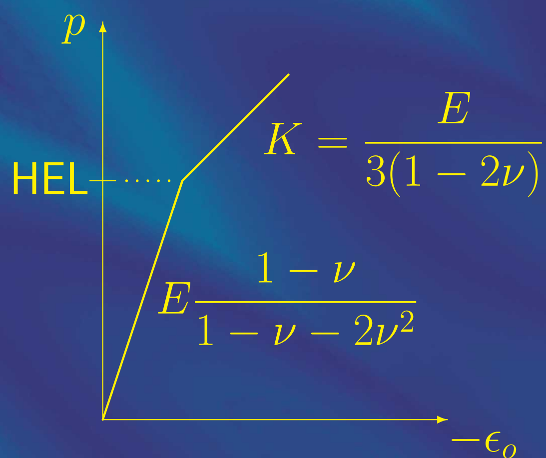
$$\boxed{-\Delta p = K\Delta\epsilon_o}$$

Hugoniot problem (3/3)



(no plastic collapse)

Hugoniot problem (3/3)



(no plastic collapse)

Hugoniot elastic limit

$$\text{HEL} = \frac{1-\nu}{1-2\nu}\sigma_Y \simeq 1.75\,\sigma_Y$$

Bulk sound speed

$$c_0 = \sqrt{\frac{K}{\rho_0}}$$

Hugoniot empirical relation

$c = c_0 + sv$ (for most metals: $s \simeq 1.5$)

Remark: Used in the hydrodynamic approximation of shock waves.

Taylor & Quinney (1931)

Lode's parameter

$$\mu_\sigma = \frac{\sigma_2 - (\sigma_1 + \sigma_3)/2}{(\sigma_1 - \sigma_3)/2}$$

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$$\mu_\sigma = \frac{\sigma_2 - (\sigma_1 + \sigma_3)/2}{(\sigma_1 - \sigma_3)/2} = \frac{2\sigma_2 - (\sigma_1 + \sigma_3)}{\sigma_1 - \sigma_3}$$

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Analogically

$$\mu_{\dot{\epsilon}^p} = \frac{3\dot{\epsilon}_2^p}{\dot{\epsilon}_1^p - \dot{\epsilon}_3^p}$$

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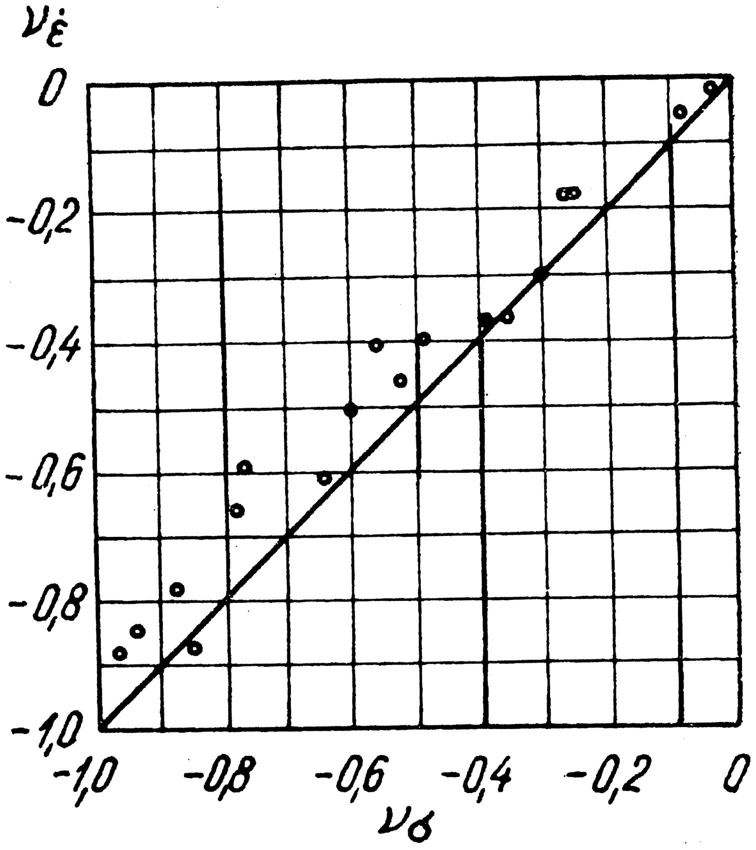
Analogically

$$\mu_{\dot\epsilon^p} = \frac{3\dot\epsilon_2^p}{\dot\epsilon_1^p - \dot\epsilon_3^p} = \frac{3S_2}{S_1 - S_3} = \mu_\sigma$$

Thus, we need to test

$$\boxed{\mu_{\dot\epsilon^p} = \mu_\sigma \quad \text{for } \mu \in [-1, 0]}$$



Experimentální ověření závislosti

$$\nu_\sigma = \nu_{\dot{\varepsilon}}$$

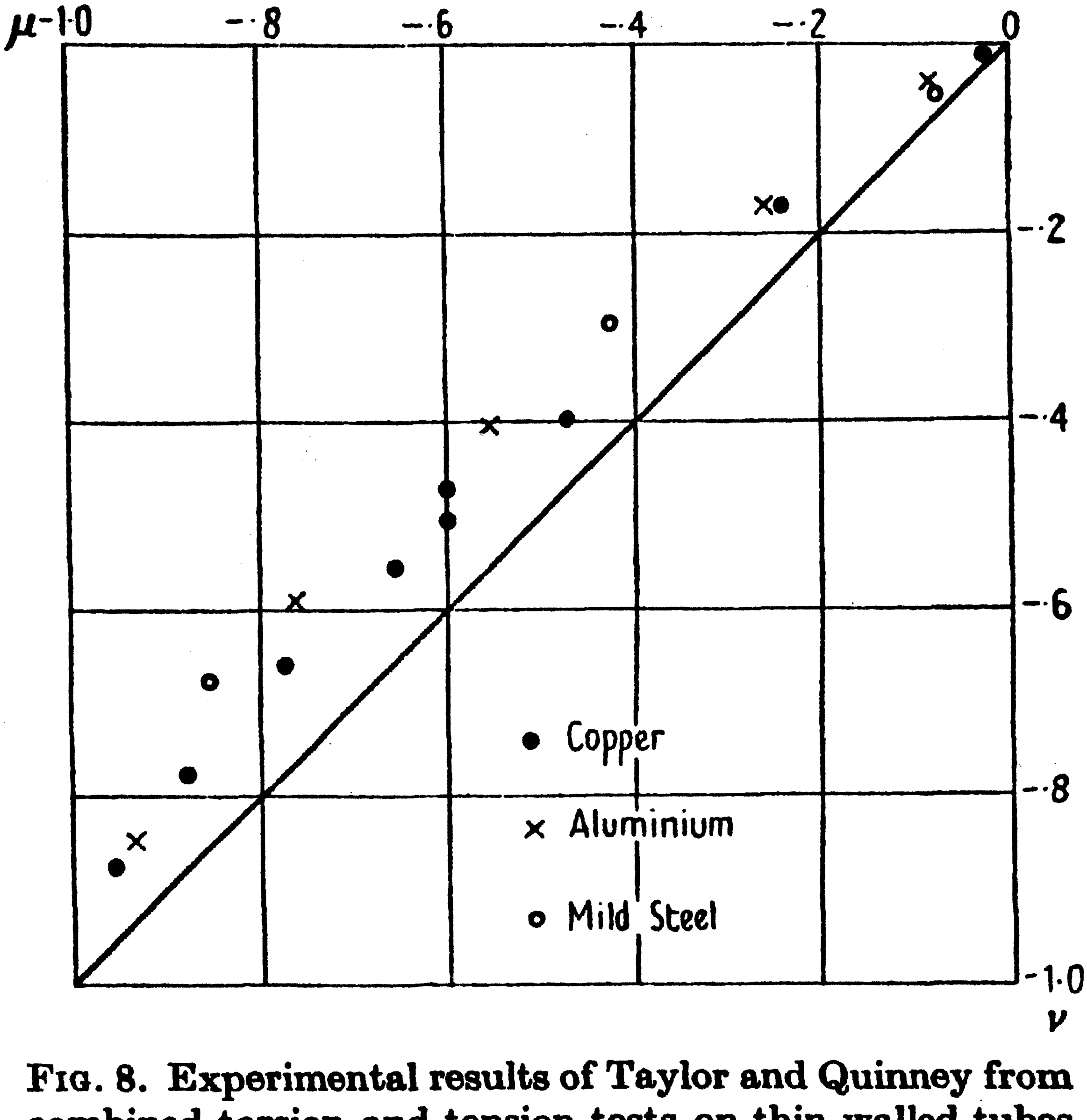


Fig. 8. Experimental results of Taylor and Quinney from combined torsion and tension tests on thin-walled tubes (μ and ν are Lode's variables).

Lode & Nádai (1928)

Proportional loading

$$\mathbf{S}(t) = g(t)\mathbf{S}^0$$

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Prandtl-Reuss equation

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Integration

$$\boldsymbol{\epsilon}^p = h(t)\mathbf{S}^0 = \frac{h(t)}{g(t)}\mathbf{S} = \lambda_H \mathbf{S} \qquad \text{(Hencky, 1924)}$$

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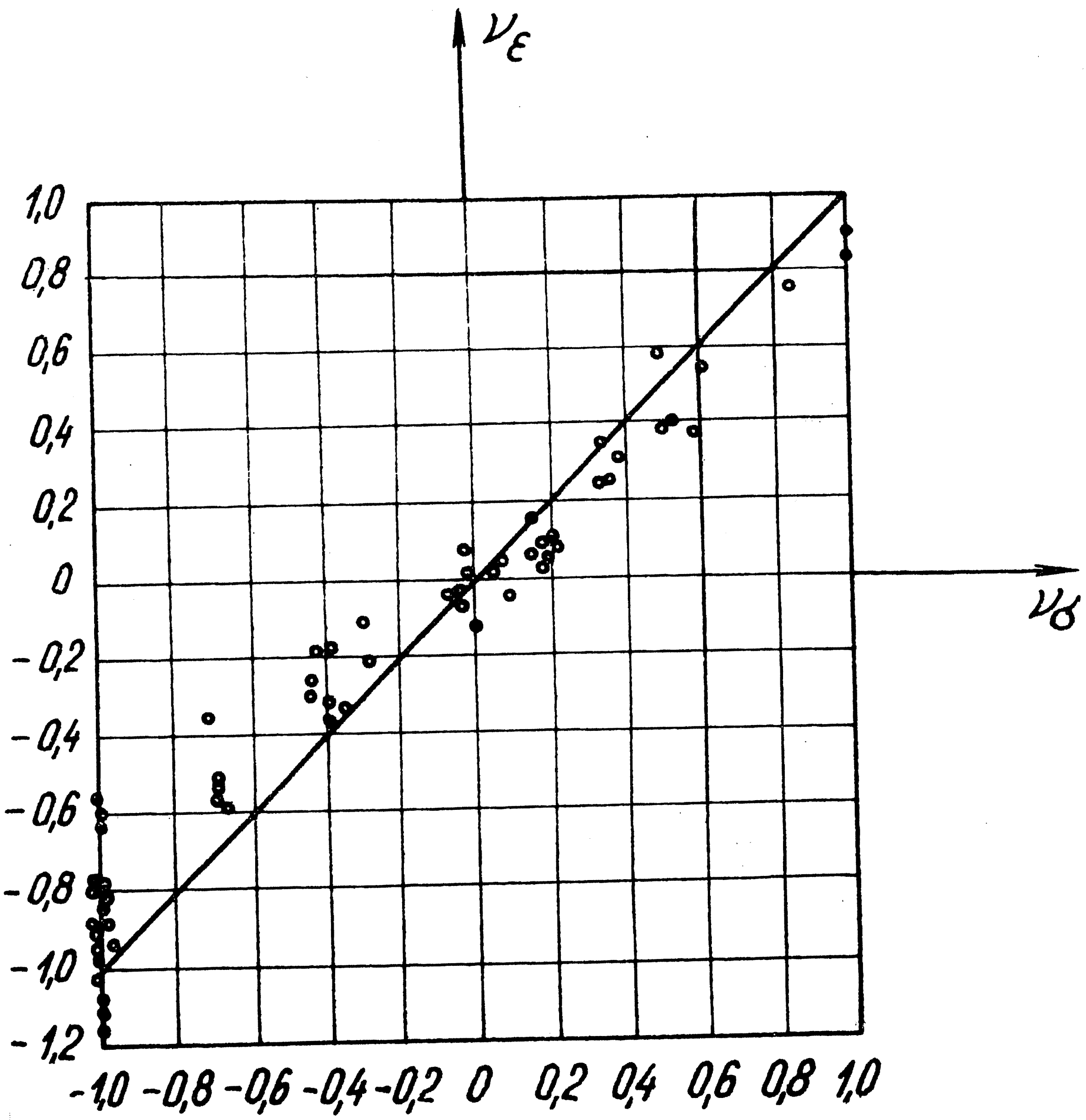
Therefore

$$\boxed{\mu_{\epsilon^p} = \mu_\sigma \qquad \text{for } \mu \in [-1, 1]}$$



Experimentální ověření závislosti

$$\nu_\sigma = \nu_\varepsilon$$

Discussion of the controversy

Assuming linearity of the flow rule

$$\dot{\boldsymbol{\epsilon}}^p = \lambda \mathcal{L}(\boldsymbol{\sigma})$$

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Assuming isotropy

$$\begin{aligned}
\dot{\epsilon}_1^p &= A\sigma_1 + B\sigma_2 + B\sigma_3 \\
\dot{\epsilon}_2^p &= B\sigma_1 + A\sigma_2 + B\sigma_3 \\
\dot{\epsilon}_3^p &= B\sigma_1 + B\sigma_2 + A\sigma_3
\end{aligned}$$

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Mean strain

$$\dot{\epsilon}_m^p = A\sigma_m + 2B\sigma_m$$

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Assuming incompressibility

$$2B = -A$$

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\end{aligned}$$

Mean strain

$$\dot{\epsilon}_m^p = A\sigma_m + 2B\sigma_m$$

Assuming incompressibility

$$2B = -A$$

Thus, we have

$$\begin{aligned}
2\dot{\epsilon}_1^p &= A(2\sigma_1 - (\sigma_2 + \sigma_3)) \\
2\dot{\epsilon}_2^p &= A(2\sigma_2 - (\sigma_1 + \sigma_3)) \\
2\dot{\epsilon}_3^p &= A(2\sigma_3 - (\sigma_1 + \sigma_2))
\end{aligned}$$

Discussion of the controversy

Assuming linearity of the flow rule

$$\dot{\boldsymbol{\epsilon}}^p = \lambda \mathcal{L}(\boldsymbol{\sigma})$$

Assuming isotropy

$$\begin{aligned} \dot{\epsilon}_1^p &= A\sigma_1 + B\sigma_2 + B\sigma_3 \\ \dot{\epsilon}_2^p &= B\sigma_1 + A\sigma_2 + B\sigma_3 \\ \dot{\epsilon}_3^p &= B\sigma_1 + B\sigma_2 + A\sigma_3 \end{aligned}$$

Mean strain

$$\dot{\epsilon}_m^p = A\sigma_m + 2B\sigma_m$$

Assuming incompressibility

$$2B = -A$$

Thus, we have

$$\begin{aligned} 2\dot{\epsilon}_1^p &= A(2\sigma_1 - (\sigma_2 + \sigma_3)) \\ 2\dot{\epsilon}_2^p &= A(2\sigma_2 - (\sigma_1 + \sigma_3)) \\ 2\dot{\epsilon}_3^p &= A(2\sigma_3 - (\sigma_1 + \sigma_2)) \end{aligned}$$

Deviator completion

$$\begin{aligned} 2\dot{\epsilon}_1^p &= 3AS_1 \\ 2\dot{\epsilon}_2^p &= 3AS_2 \\ 2\dot{\epsilon}_3^p &= 3AS_3 \end{aligned}$$

Discussion of the controversy

Assuming linearity of the flow rule

$$\dot{\boldsymbol{\epsilon}}^p = \lambda \mathcal{L}(\boldsymbol{\sigma})$$

Assuming isotropy

$$\begin{aligned}
\dot{\epsilon}_1^p &= A\sigma_1 + B\sigma_2 + B\sigma_3 \\
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Therefore

$$\boxed{\dot{\boldsymbol{\epsilon}}^p = \lambda_p \mathbf{S}}$$

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Therefore

$$\boxed{\dot{\boldsymbol{\epsilon}}^p = \lambda_p \mathbf{S}}$$

Conclusion: Slight non-linearity of $\mathbf{R}(\boldsymbol{\sigma})$ is probably the culprit.

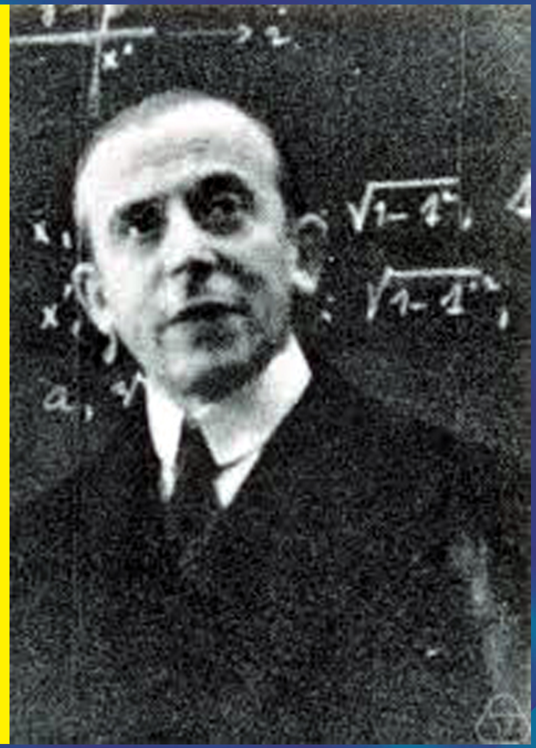
Henri Édouard Tresca

(1814-1885)

Adémar J. C. Barré de Saint-Vénant

(1797-1886)

Richard Edler von Mises



(1883-1953)