Institute of Thermomechanics
ASCR

LIMIT LOAD ANALYSIS II

Jiří Plešek

Institute of Thermomechanics
Czech Academy of Sciences

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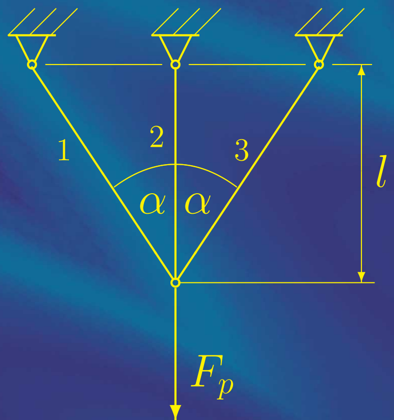
  - cantilever
- Computing dissipation
  - thick wall tube
  - plate with a hole

Truss structure



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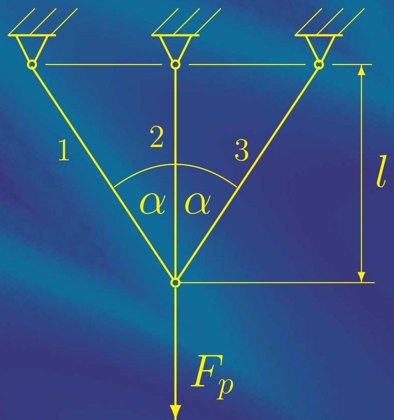
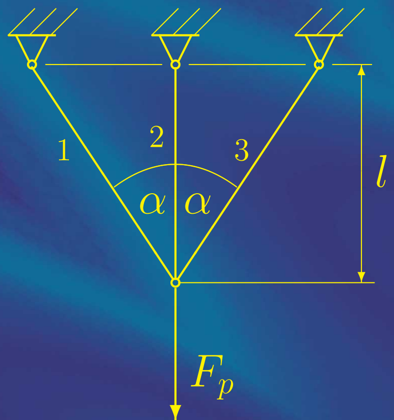


Admissible stress method

SAS: all $N_i = N_Y$

Truss structure



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$$\boxed{F_p \geq N_Y(1 + 2\cos\alpha)}$$

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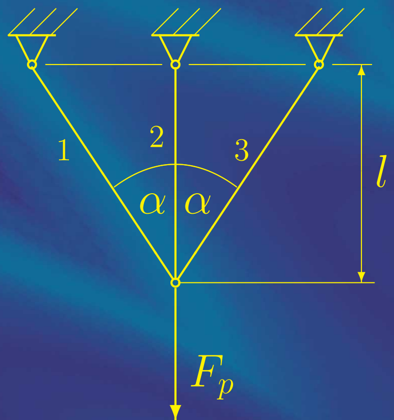

Principle of virtual work

KAS: $(\Delta l_2)^{\cdot} = v$

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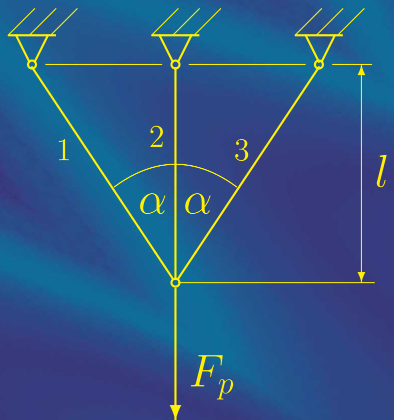

Principle of virtual work
KAS: $(\Delta l_2)^{\cdot} = v$

Compatibility

$$(\Delta l_1)^{\cdot} = (\Delta l_2 \cos\alpha)^{\cdot} = v\cos\alpha$$

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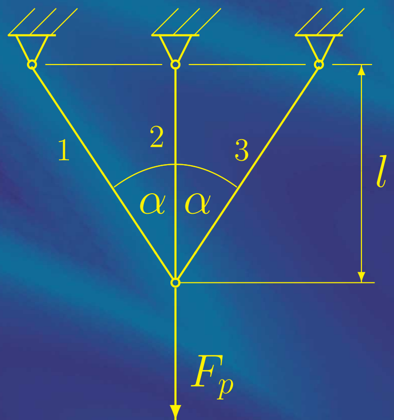

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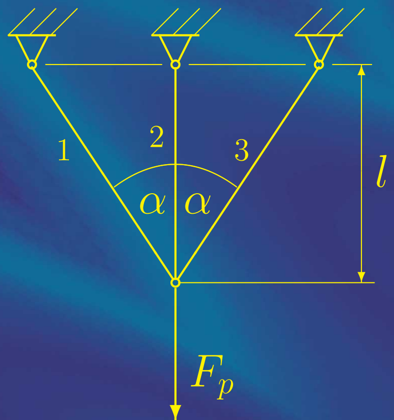
Lemma: all truss flow, therefore

$$\mathcal{D} = N_Y v + 2 N_Y v\, cos \alpha$$

Truss structure



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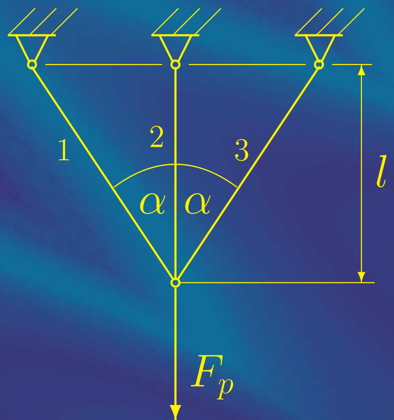
$$\boxed{F_p \leq N_Y(1 + 2\cos\alpha)}$$

Truss structure



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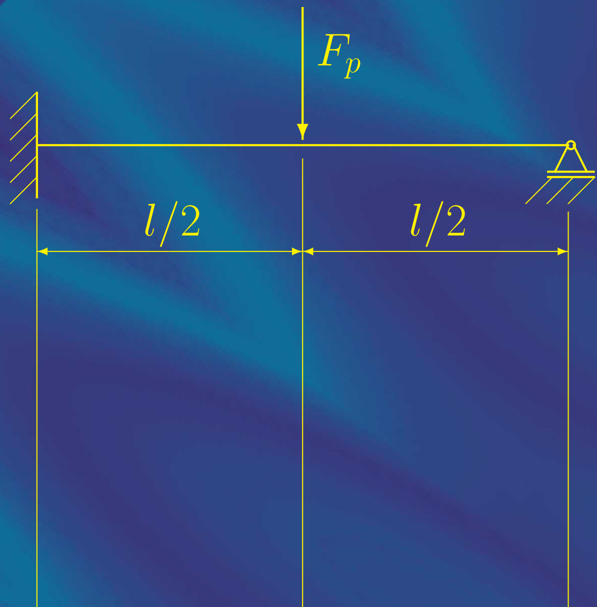
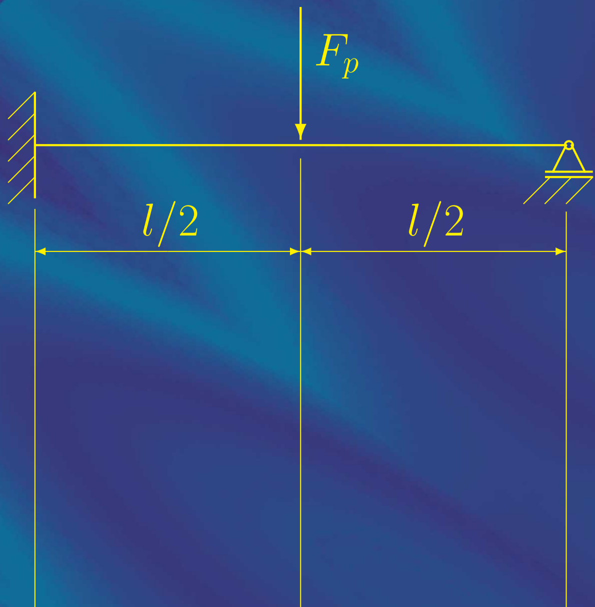
Remark: Either equilibrium or compatibility equations used but not both.

Cantilever

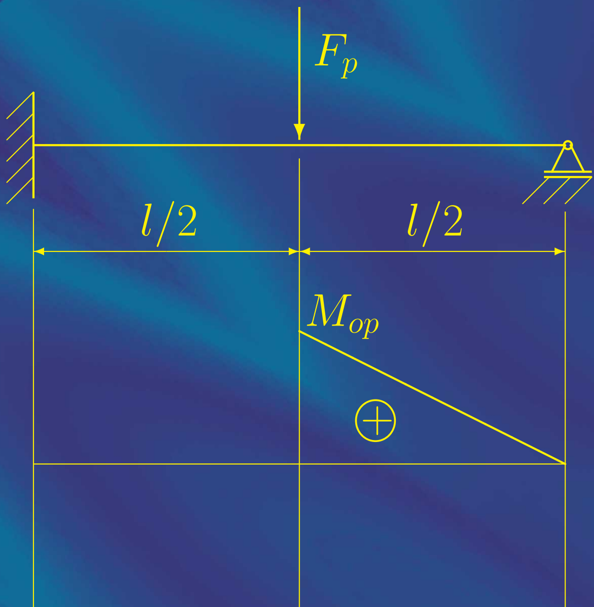


Cantilever



PVW: $$\boxed{F_p \leq \frac{6M_{op}}{l}}$$

Cantilever

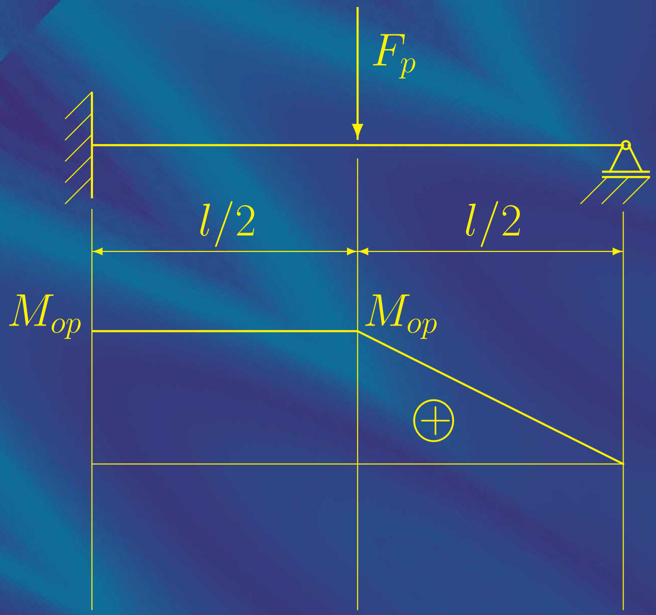


Right support reaction

$$\frac{Rl}{2} = M_{op} \quad \Rightarrow \quad R = \frac{2M_{op}}{l}$$

PVW: $$\boxed{F_p \leq \frac{6M_{op}}{l}}$$

Cantilever



Right support reaction

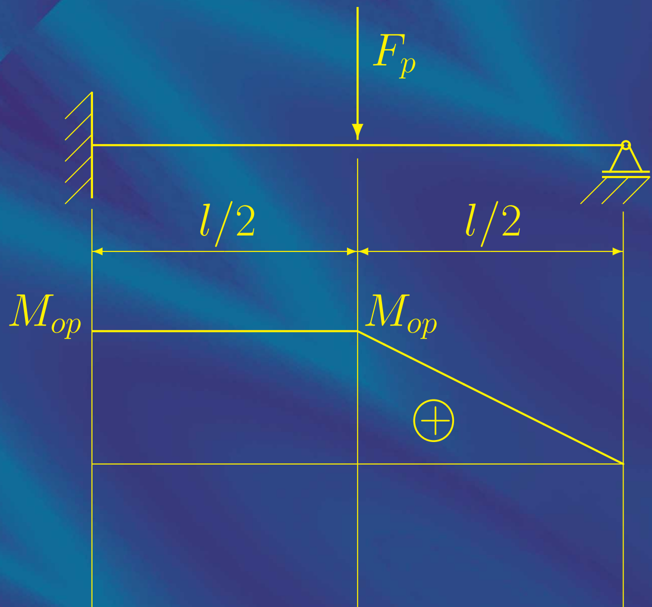
$$\frac{Rl}{2} = M_{op} \quad \Rightarrow \quad R = \frac{2M_{op}}{l}$$

Equilibrium equation

$$Rl - \frac{F_p l}{2} - M_{op} = 0$$

PVW: $\boxed{F_p \leq \dfrac{6M_{op}}{l}}$

Cantilever



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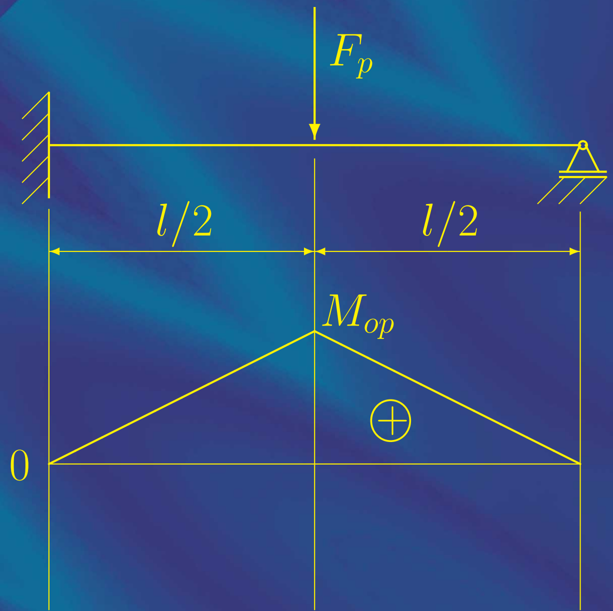
Cantilever



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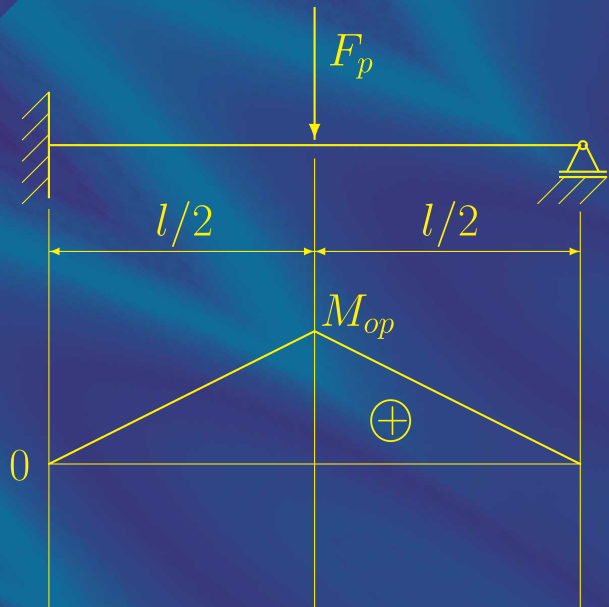
Equilibrium equation

$$Rl - \frac{F_p l}{2} + 0 = 0$$

PVW: $\boxed{F_p \leq \frac{6M_{op}}{l}}$

Cantilever



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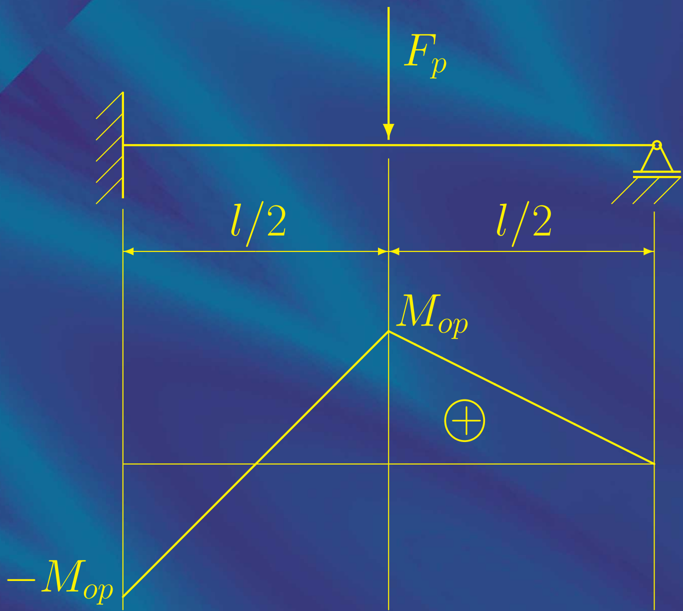
Equilibrium equation

$$Rl - \frac{F_p l}{2} + 0 = 0$$

ASM: $\boxed{F_p \geq \frac{4M_{op}}{l}}$

PVW: $\boxed{F_p \leq \frac{6M_{op}}{l}}$

Cantilever



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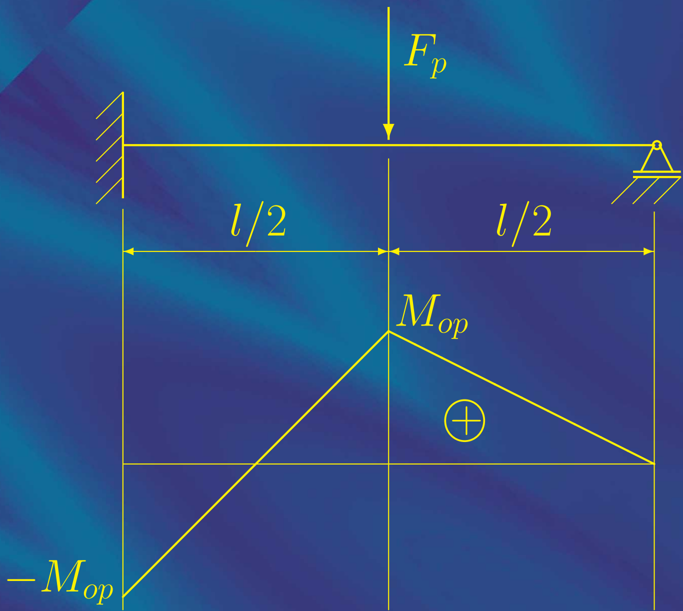

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Equilibrium equation

$$Rl - \frac{F_p l}{2} + M_{op} = 0$$

ASM: $\boxed{F_p \geq \frac{6M_{op}}{l}}$

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Tresca's dissipation

Yield condititon

$$\tau_e = \sigma_1 - \sigma_3 = \sigma_Y$$

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Associated flow rule

$$\begin{aligned} \dot{\epsilon}_1^p &= +\lambda \\ \dot{\epsilon}_2^p &= 0 \\ \dot{\epsilon}_3^p &= -\lambda \end{aligned}$$

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Dissipation

$$\boxed{\mathcal{D} = \int_{\Omega_p} \dot{w}_p \,\mathrm{d}V = \int_{\Omega_p} \sigma_Y \dot{\epsilon}_1^p \,\mathrm{d}V}$$

Thick wall tube

KAS: principal strain rates: $\dot{\epsilon}_t > 0$, $\dot{\epsilon}_o = 0$, $\dot{\epsilon}_r < 0$

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Using the Lemma and Tresca flow rule

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Solution

$$v = \frac{C}{x} \quad \text{and} \quad \dot{\epsilon}_t = \frac{C}{x^2} \equiv \dot{\epsilon}_1$$

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Dissipation

$$\mathcal{D} = \int_{r_1}^{r_2} \sigma_Y \left(\frac{C}{x^2}\right) 2\pi x \, \mathrm{d}x = 2\pi C \sigma_Y \ln \frac{r_2}{r_1}$$

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PVW: $\quad \mathcal{D} = 2\pi r_1 p v$

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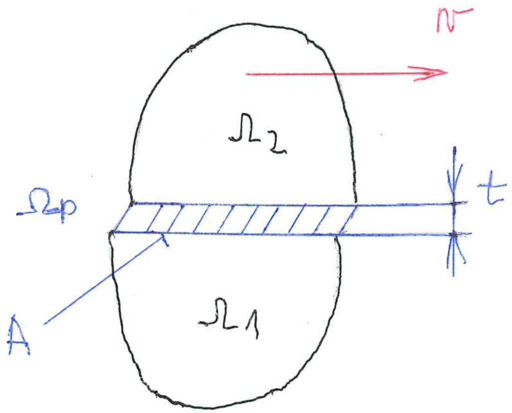
PVW: $\mathcal{D} = 2\pi r_1 p v = 2\pi C p \quad \Rightarrow \quad \boxed{p = \sigma_Y \ln \frac{r_2}{r_1}}$

Remark: The same result as for the admissible stress method—see Lecture 3.

Tresca shear band



$$\boxed{\dot{\gamma} = \frac{v}{t}}$$

Simple shear:

$$\dot{\underline{\varepsilon}} = \frac{\dot{\gamma}}{2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\dot{\varepsilon}_1 = \frac{\dot{\gamma}}{2} \qquad \dot{\varepsilon}_2 = 0 \qquad \dot{\varepsilon}_3 = -\frac{\dot{\gamma}}{2}$$

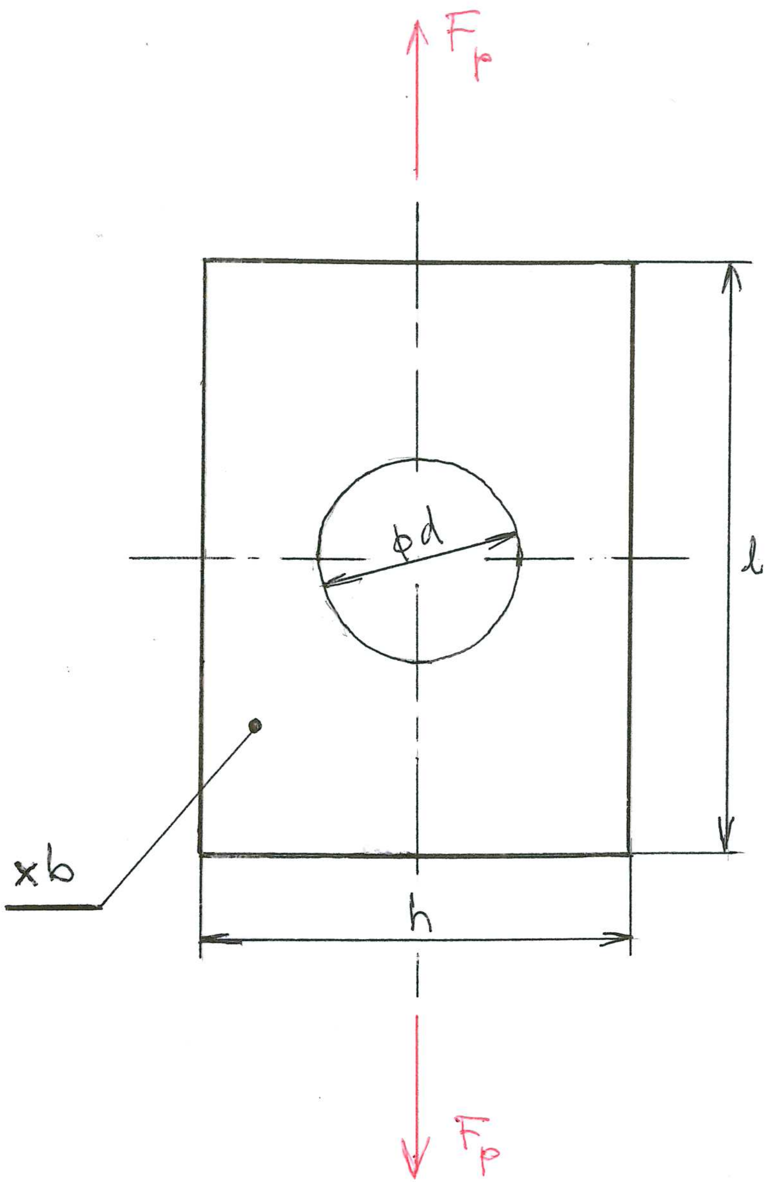
$$w_p = \sigma_Y \frac{\dot{\gamma}}{2} = \tau_Y \dot{\gamma} = \tau_Y \frac{v}{t}$$

$$\mathcal{D} = A t \tau_Y \frac{v}{t} = \underbrace{A \tau_Y}_{\text{"friction"}} v$$

(holds even for $t \to 0$)

von Mises: The exact same result with $\tau_Y = \sigma_Y / \sqrt{3}$.

Plate with a hole

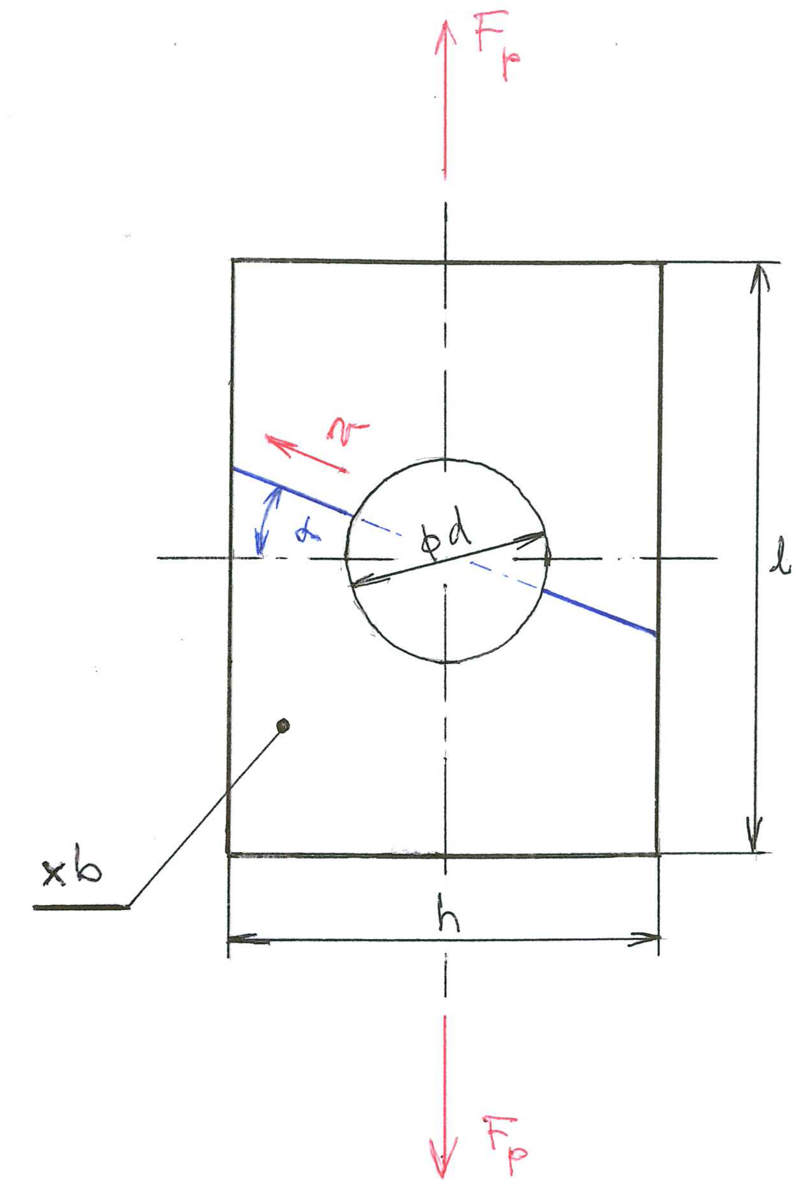


ASM

$$\boxed{F_p \geq (h-d)\, b \cdot \sigma_y}$$

(universal solution)

Plate with a hole



ASM

$$\boxed{F_p \geq (h-d)\, b \cdot \sigma_Y}$$

(universal solution)

$$D = \left(\frac{h}{\cos\alpha} - d\right) b\, \tau_Y \cdot v = F_p\, v \cdot \sin\alpha$$

PVW

$$\boxed{F_p \leq \left(\frac{h}{\cos\alpha} - d\right) \frac{b}{\sin\alpha}\, \tau_Y}$$

optimize: $\tau_Y = \dfrac{\sigma_Y}{2}$, $\dfrac{d}{h} = \dfrac{1}{3}$ => $\alpha = 41^\circ$

Note: □, $\alpha = 45^\circ$ => exact result.

Discussion

Putting it all together

$$1 - \frac{d}{h} \le \frac{F_p}{bh\sigma_Y} \le \left(\frac{1}{\cos\alpha} - \frac{d}{h}\right)\frac{1}{2\sin\alpha}$$

Discussion

Putting it all together

$$1 - \frac{d}{h} \leq \frac{\sigma_\infty^p}{\sigma_Y} \leq \left(\frac{1}{\cos\alpha} - \frac{d}{h}\right)\frac{1}{2\sin\alpha}$$

Discussion

Putting it all together

$$1 - \frac{d}{h} \leq \frac{\sigma_\infty^p}{\sigma_Y} \leq \left(\frac{1}{\cos\alpha} - \frac{d}{h}\right)\frac{1}{2\sin\alpha}$$

Exact solution

- For $d/h \to 0$, $\alpha \to 45°$ and $\sigma_\infty^p \to \sigma_Y$

Discussion

Putting it all together

$$1-\frac{d}{h}\leq\frac{\sigma_{\infty}^{p}}{\sigma_{Y}}\leq\left(\frac{1}{\cos\alpha}-\frac{d}{h}\right)\frac{1}{2\sin\alpha}$$

Exact solution

- For $d/h\to 0$, $\alpha\to 45^{\circ}$ and $\sigma_{\infty}^{p}\to\sigma_{Y}$
- For $d/h\to 1$, $\alpha\to 0^{\circ}$ and $\sigma_{\infty}^{p}\to 0$

Discussion

Putting it all together

$$1 - \frac{d}{h} \leq \frac{\sigma_\infty^p}{\sigma_Y} \leq \left(\frac{1}{\cos\alpha} - \frac{d}{h}\right)\frac{1}{2\sin\alpha}$$

Exact solution

- For $d/h \to 0$, $\alpha \to 45^\circ$ and $\sigma_\infty^p \to \sigma_Y$
- For $d/h \to 1$, $\alpha \to 0^\circ$ and $\sigma_\infty^p \to 0$

Setting $d/h = 1/3$ and $\alpha = 41^\circ$

$$0.67 \leq \frac{\sigma_\infty^p}{\sigma_Y} \leq 0.76$$

Then the maximum error is (0.76 - 0.67)/0.67=13%.

FEM verification

Correction for the von Mises criterion

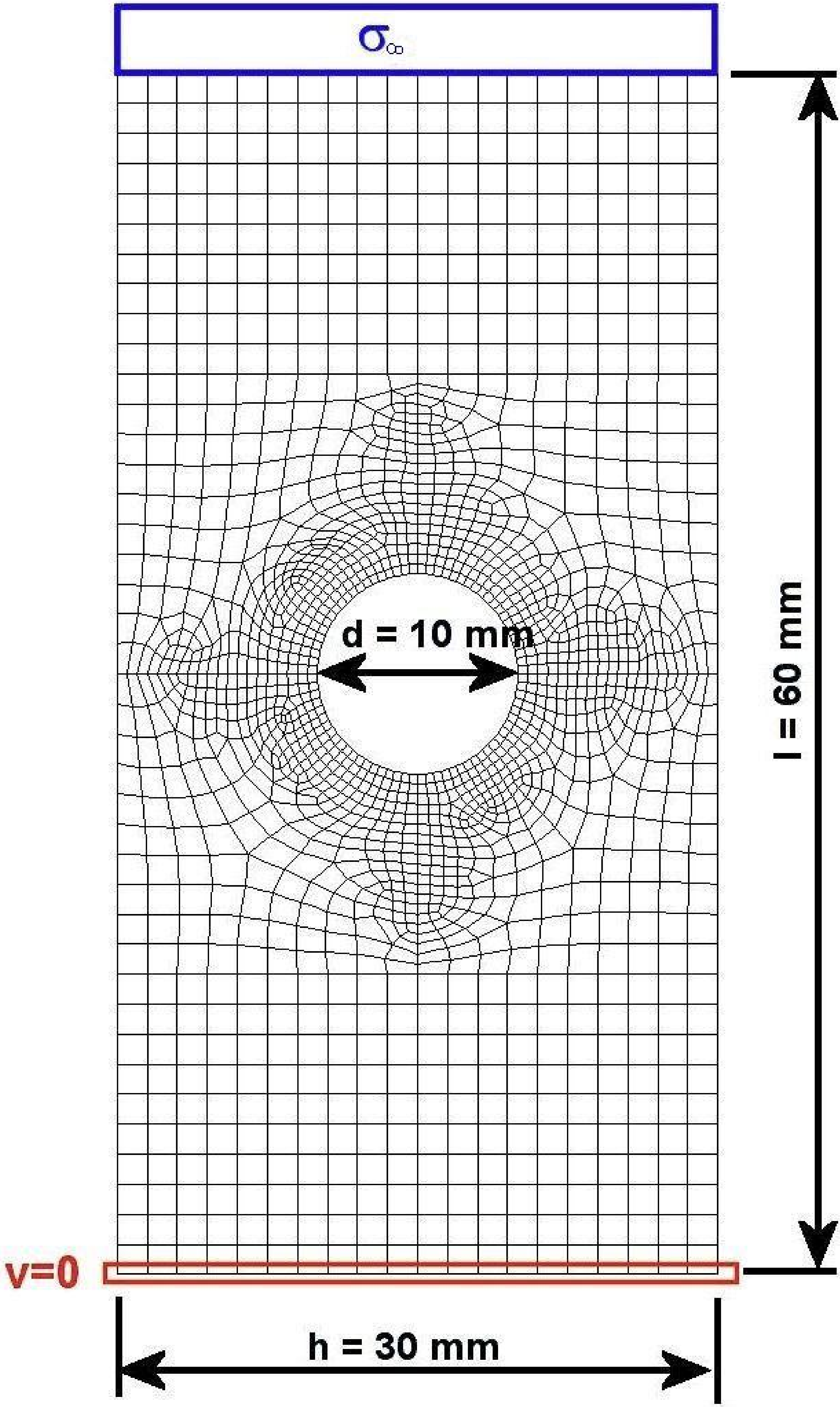
$$0.67 \leq \frac{\sigma_{\infty}^{p}}{\sigma_{Y}} \leq 0.76\frac{2}{\sqrt{3}} = 0.87$$

FEM verification

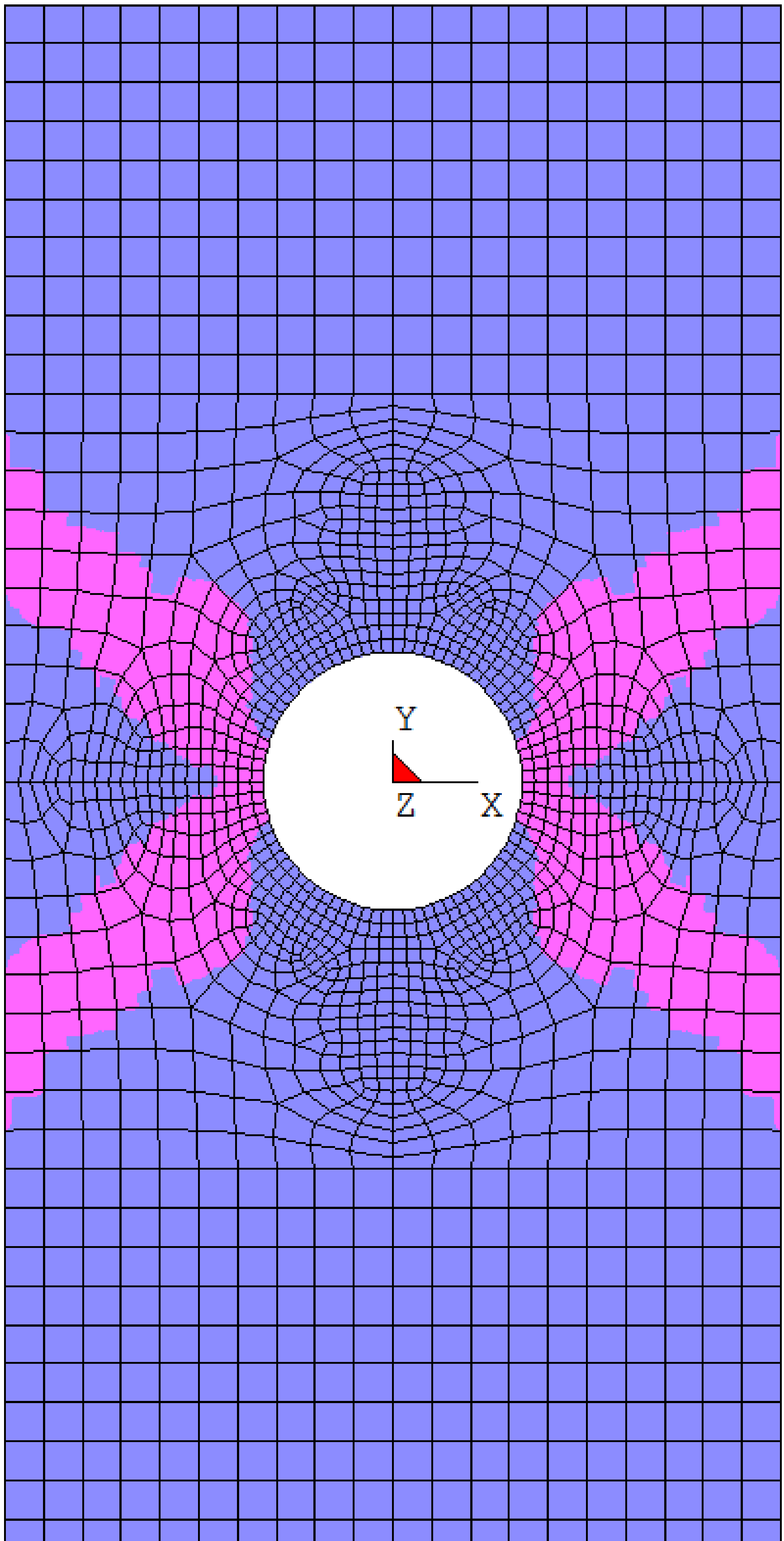
Correction for the von Mises criterion

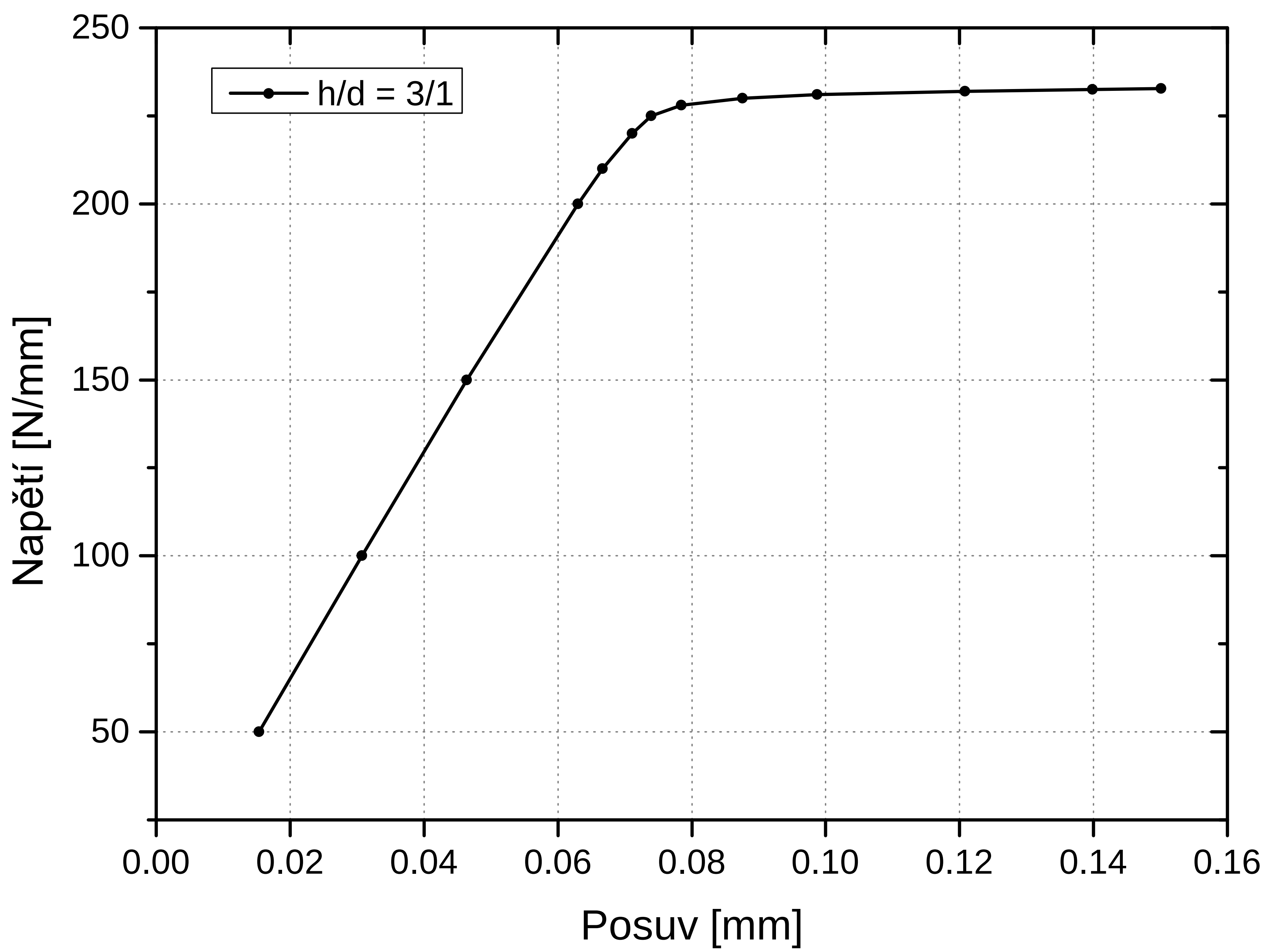
$$0.67 \leq \frac{\sigma_\infty^p}{\sigma_Y} \leq 0.76\frac{2}{\sqrt{3}} = 0.87$$

Numerical data: $E = 210$ GPa, $\nu = 0.3$, $\sigma_Y = 300$ MPa.



σ_∞
d = 10 mm
l = 60 mm
v=0
h = 30 mm


Y
Z
X



h/d = 3/1
Napětí [N/mm]
Posuv [mm]
250
200
150
100
50
0.00
0.02
0.04
0.06
0.08
0.10
0.12
0.14
0.16

FEM verification

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$$0.67 \leq \frac{\sigma_\infty^p}{\sigma_Y} \leq 0.76\frac{2}{\sqrt{3}} = 0.87$$

Numerical data: $E = 210\,$GPa, $\nu = 0.3$, $\sigma_Y = 300\,$MPa.

Read from the plot: $\sigma_\infty^p = 232\,$MPa.

FEM verification

Correction for the von Mises criterion

$$0.67 \leq \frac{\sigma_\infty^p}{\sigma_Y} \leq 0.76\frac{2}{\sqrt{3}} = 0.87$$

Numerical data: $E = 210\,\mathrm{GPa}$, $\nu = 0.3$, $\sigma_Y = 300\,\mathrm{MPa}$.

Read from the plot: $\sigma_\infty^p = 232\,\mathrm{MPa}$.

$$\boxed{\frac{\sigma_\infty^p}{\sigma_Y} = \frac{232}{300} = 0.77}$$