Institute of Thermomechanics ASCR

# YIELDING FUNCTIONS II


Jiří Plešek

Institute of Thermomechanics
Czech Academy of Sciences

# Contents

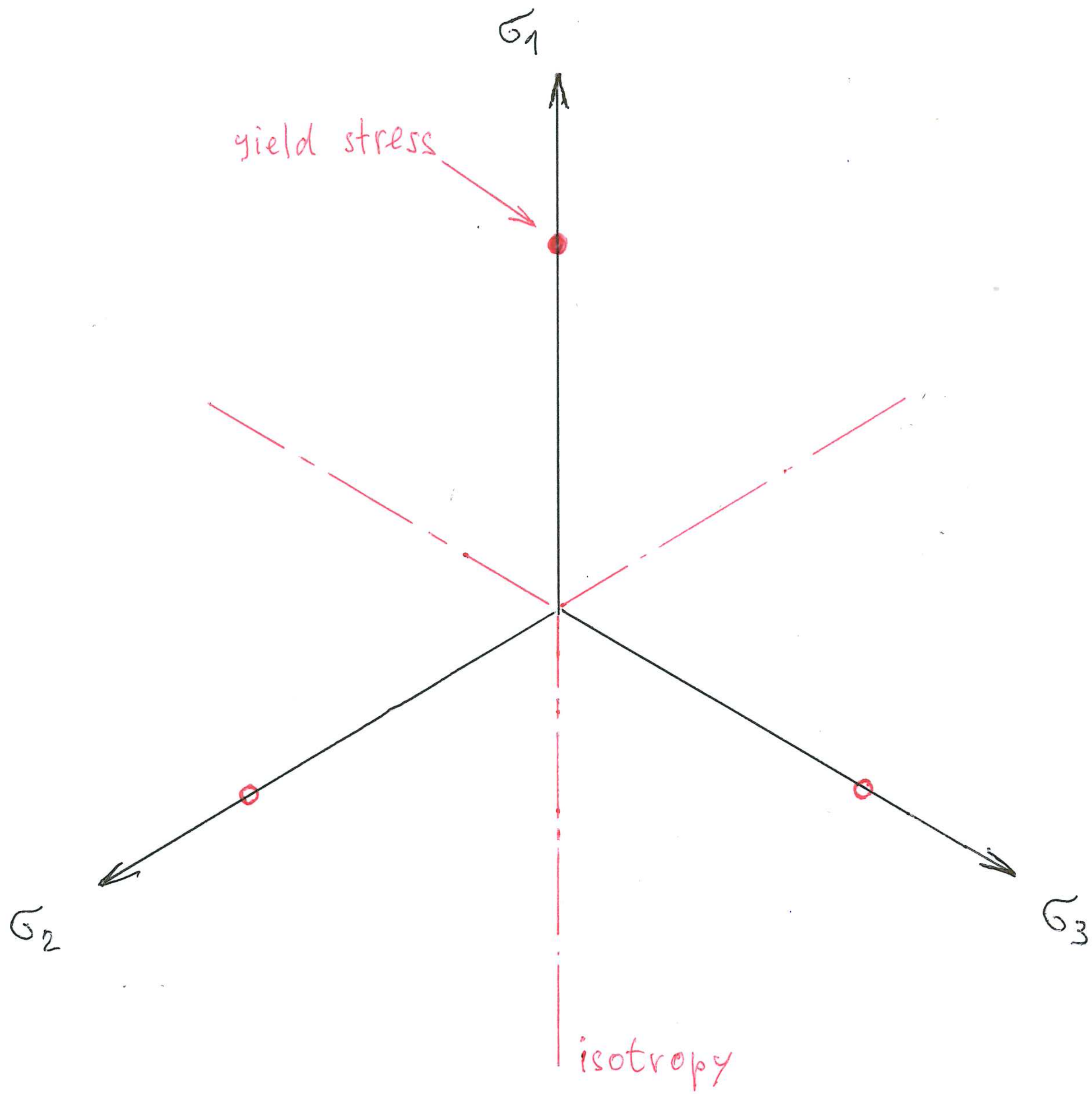
- Review
- Deviatoric plane
- Famous experimental setups
  - Lode-Nádai (1926)
  - Taylor-Quinney (1931)

# Review

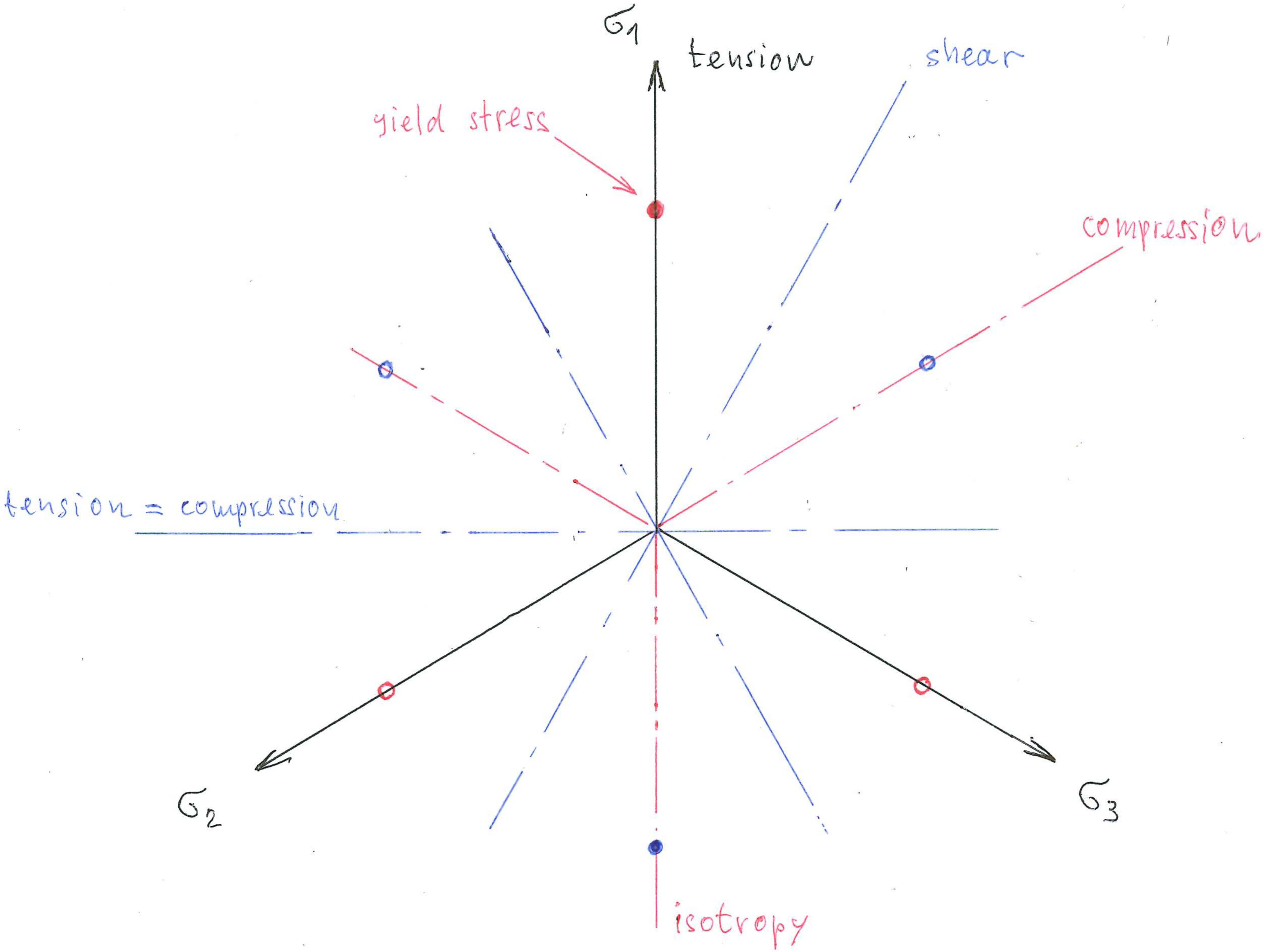
Properties of yielding functions

- Convex (always)
- Deviatoric (often)
- Tension $=$ compression (metals)
- Isotropy (often)

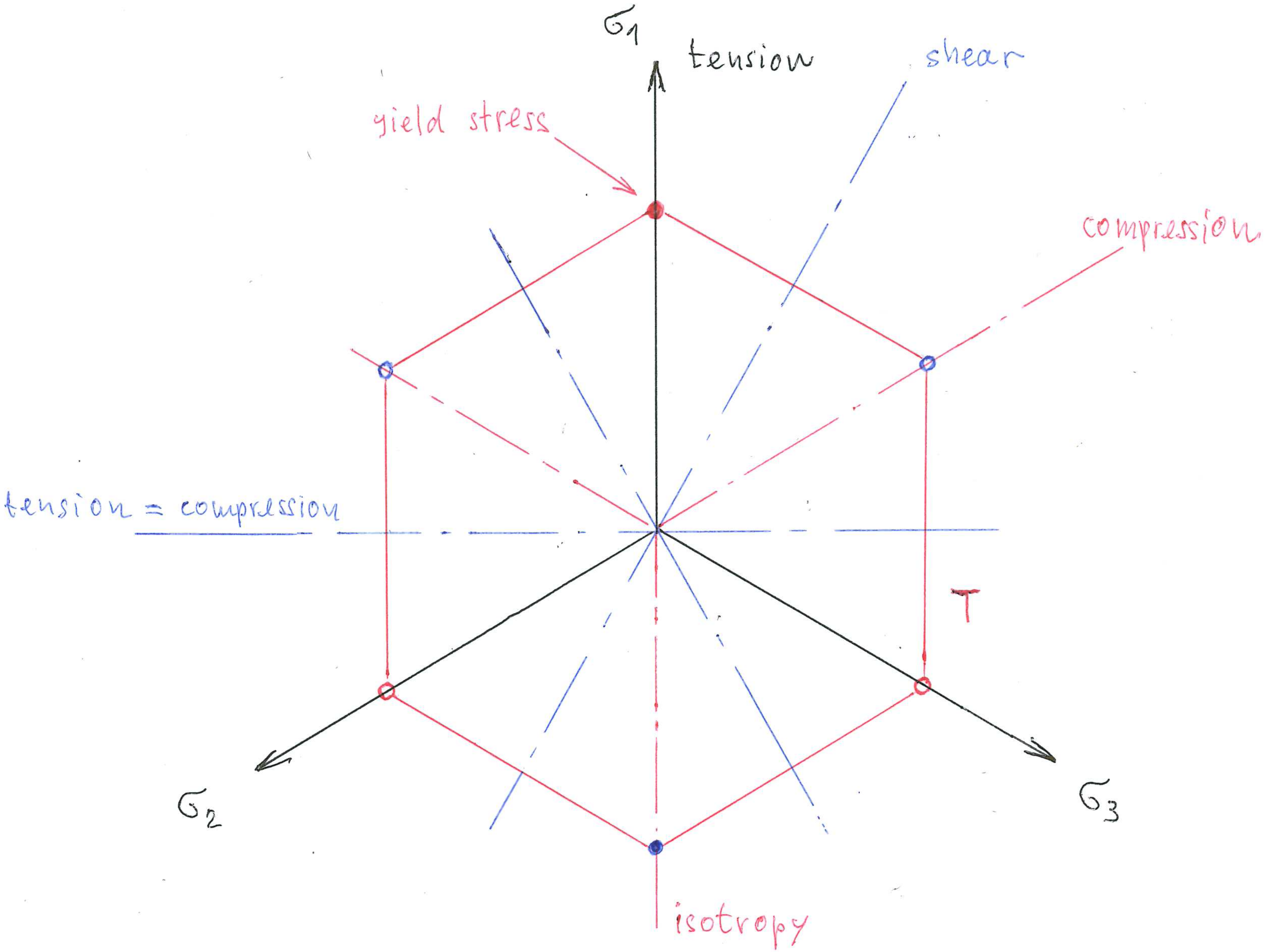
# Deviatoric plane



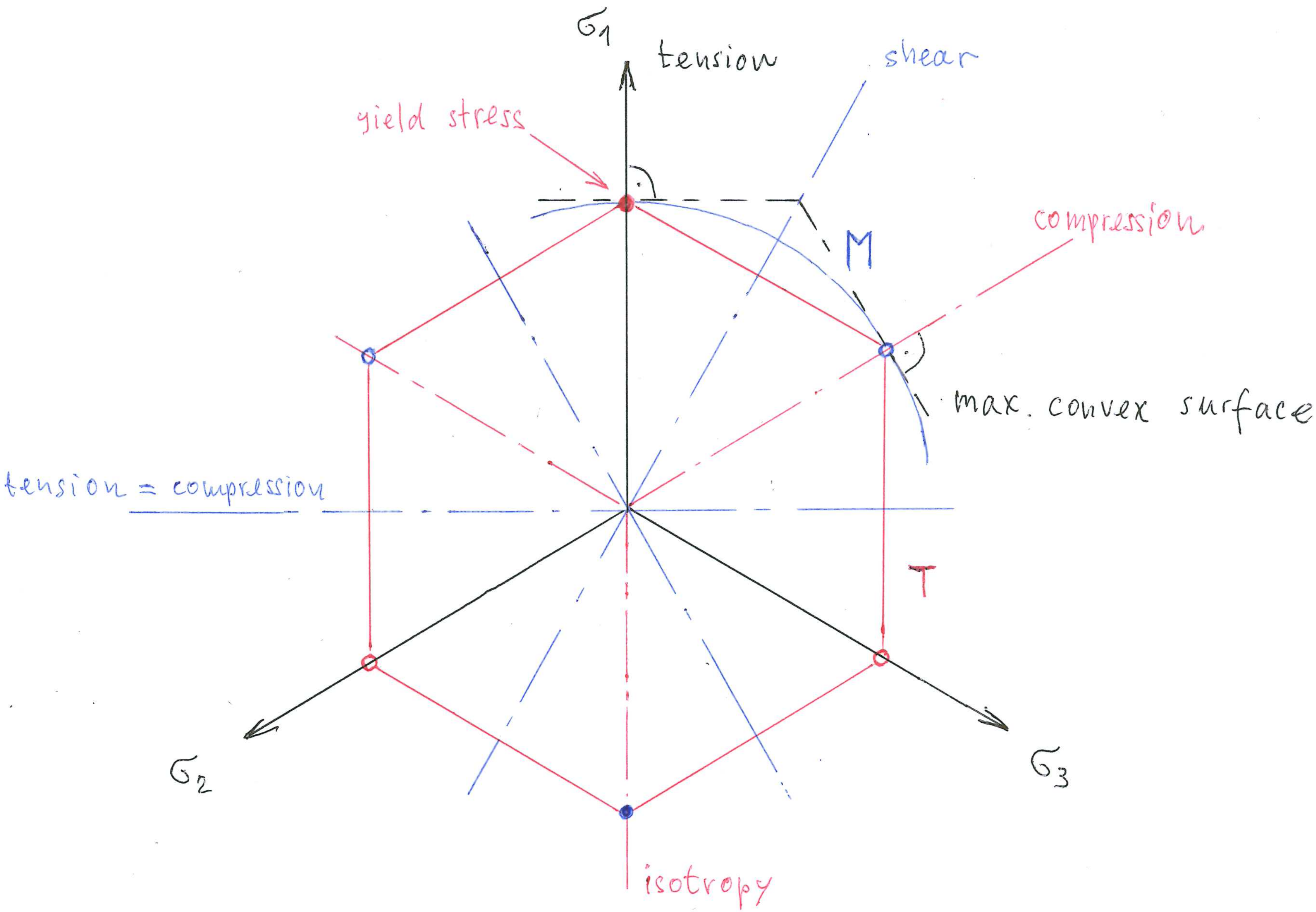
# Deviatoric plane



# Deviatoric plane

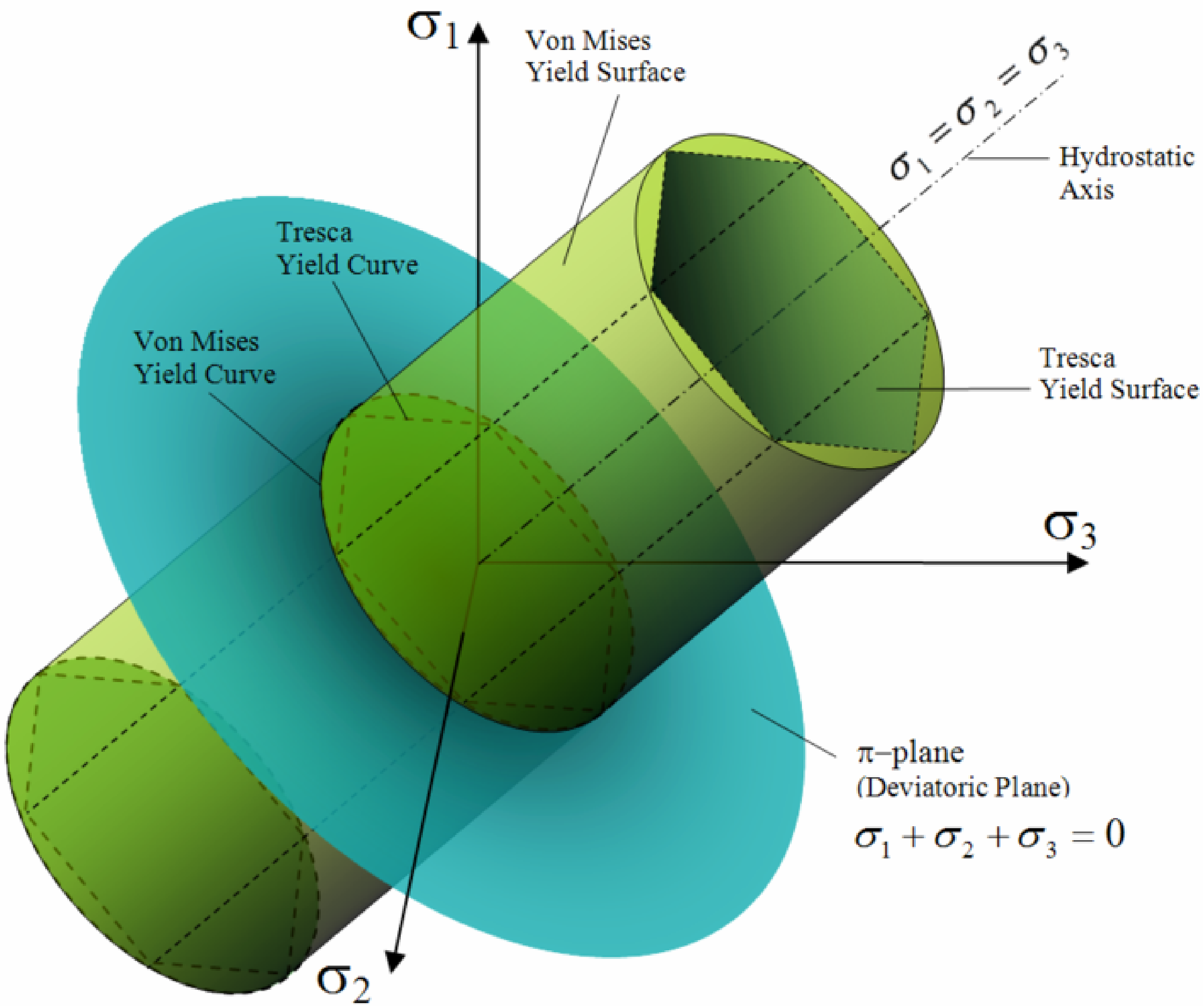


## Deviatoric plane



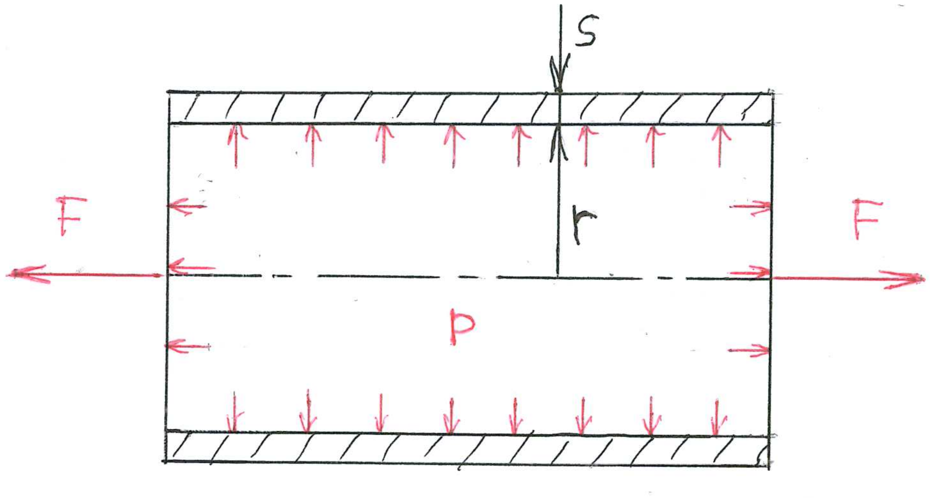

**Remark:** Tresca's condition is conservative.

# Tresca vs. von Mises




http://en.wikipedia.org/wiki/Image:Yield_surf

# Lode & Nādai (1926)

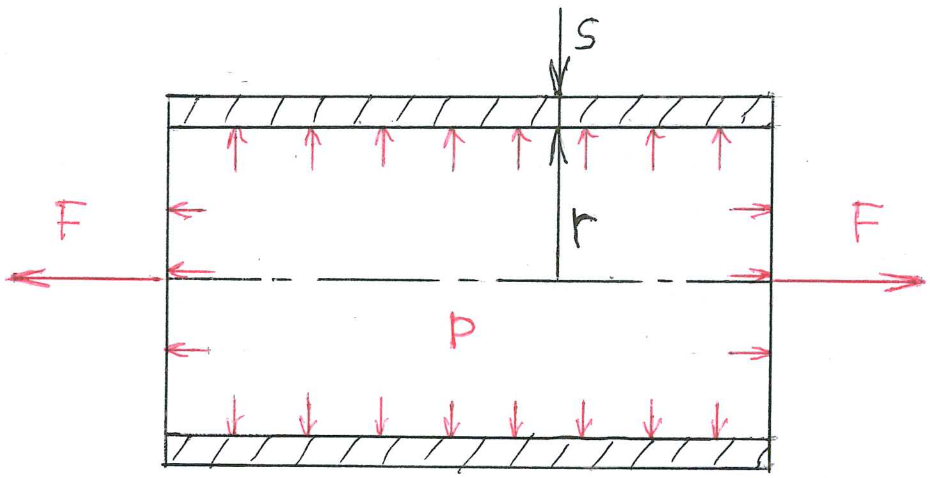



$$\sigma_t = \frac{pr}{s}$$

$$\sigma_0 = \frac{pr}{2s} + \frac{F}{2\pi r s}$$

$\sigma_r = 0$ (at surface)

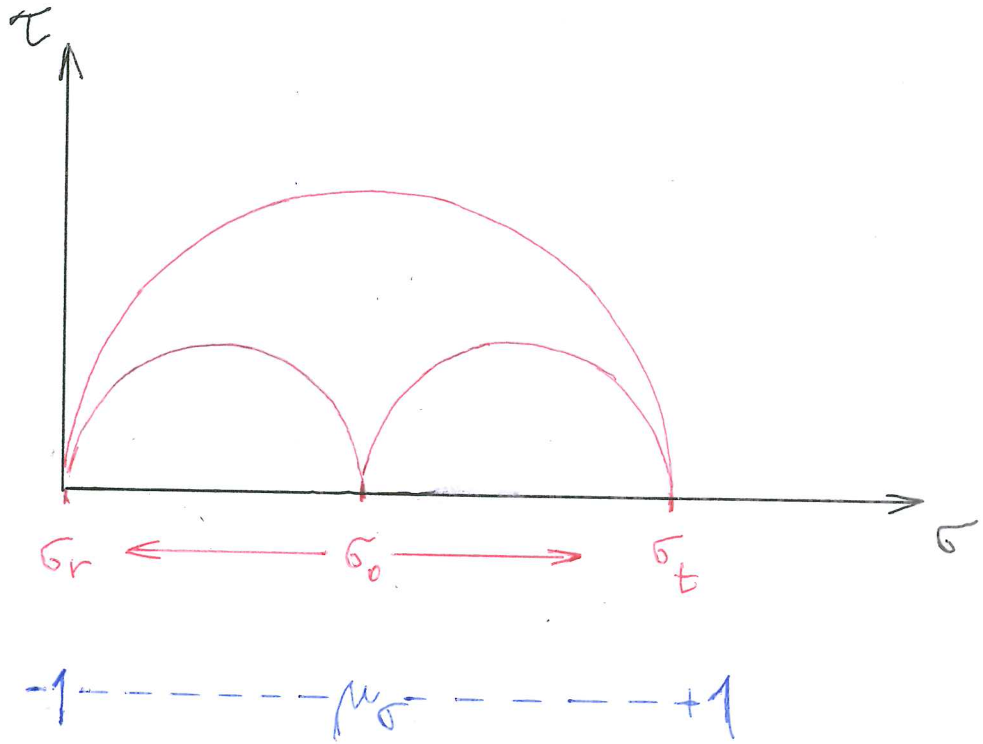
Lode & Nādai (1926)




$$\sigma_t = \frac{pr}{s}$$

$$\sigma_o = \frac{pr}{2s} + \frac{F}{2\pi r s}$$

$\sigma_r = 0$ (at surface)




$\mu_\sigma$ = Lode's parameter

(Experiments cover the whole sextant.)

# Lode's parameter

Definition

$$\mu_\sigma = \frac{\sigma_2 - (\sigma_1 + \sigma_3)/2}{(\sigma_1 - \sigma_3)/2} \quad \in \quad [-1, 1]$$

# Lode's parameter

Definition

$$\mu_\sigma = \frac{\sigma_2 - (\sigma_1 + \sigma_3)/2}{(\sigma_1 - \sigma_3)/2} \quad \in \quad [-1, 1]$$

Quizz

- $\mu_\sigma = -1$
- $\mu_\sigma = \quad 0$
- $\mu_\sigma = +1$

# Lode's parameter

Definition

$$\mu_\sigma = \frac{\sigma_2 - (\sigma_1 + \sigma_3)/2}{(\sigma_1 - \sigma_3)/2} \quad \in \quad [-1, 1]$$

Quizz

- $\mu_\sigma = -1$ tension
- $\mu_\sigma = \phantom{+}0$
- $\mu_\sigma = +1$

# Lode's parameter

Definition

$$\mu_\sigma = \frac{\sigma_2 - (\sigma_1 + \sigma_3)/2}{(\sigma_1 - \sigma_3)/2} \quad \in \quad [-1, 1]$$

Quizz

- $\mu_\sigma = -1$ tension
- $\mu_\sigma = \ \ 0$ shear
- $\mu_\sigma = +1$

# Lode's parameter

Definition

$$\mu_\sigma = \frac{\sigma_2 - (\sigma_1 + \sigma_3)/2}{(\sigma_1 - \sigma_3)/2} \quad \in \quad [-1, 1]$$

Quizz

- $\mu_\sigma = -1$ tension
- $\mu_\sigma = \phantom{+}0$ shear
- $\mu_\sigma = +1$ compression

# Lode's parameter

Definition

$$\mu_\sigma = \frac{\sigma_2 - (\sigma_1 + \sigma_3)/2}{(\sigma_1 - \sigma_3)/2} \quad \in \quad [-1, 1]$$

Quizz

- $\mu_\sigma = -1$ tension
- $\mu_\sigma = \ \ 0$ shear
- $\mu_\sigma = +1$ compression

Exercise: Which of $\sigma_r$, $\sigma_o$, $\sigma_t$ causes tension/shear/compression?

# Lode's parameter

Definition

$$\mu_\sigma = \frac{\sigma_2 - (\sigma_1 + \sigma_3)/2}{(\sigma_1 - \sigma_3)/2} \quad \in \quad [-1, 1]$$

Quizz

- $\mu_\sigma = -1$ tension
- $\mu_\sigma = \phantom{+}0$ shear
- $\mu_\sigma = +1$ compression

Exercise: Which of $\sigma_r$, $\sigma_o$, $\sigma_t$ causes tension/shear/compression?

Remark: Biaxial tension generates uniaxial pressure!

Note: Any stress state is covered by varying $\mu_\sigma$.

# Prediction of yielding

Tresca

$$\tau_e = \sigma_1 - \sigma_3 \leq \sigma_Y$$

(independent of $\mu_\sigma$)

# Prediction of yielding

Tresca

$$\tau_e = \sigma_1 - \sigma_3 \leq \sigma_Y$$

(independent of $\mu_\sigma$)

$$\boxed{\frac{\sigma_1 - \sigma_3}{\sigma_Y} \leq 1}$$

where

$$\frac{\sigma_1 - \sigma_3}{\sigma_Y} = \frac{\sigma_t}{\sigma_Y} = \frac{pr}{s\sigma_Y}$$

# Prediction of yielding

**Tresca**

$$\tau_e = \sigma_1 - \sigma_3 \leq \sigma_Y$$

(independent of $\mu_\sigma$)

$$\boxed{\frac{\sigma_1 - \sigma_3}{\sigma_Y} \leq 1}$$

where

$$\frac{\sigma_1 - \sigma_3}{\sigma_Y} = \frac{\sigma_t}{\sigma_Y} = \frac{pr}{s\sigma_Y}$$

**von Mises**

$$\sigma_2 = \mu_\sigma \frac{\sigma_1 - \sigma_3}{2} + \frac{\sigma_1 + \sigma_3}{2}$$

Institute of Thermomechanics ASCR

# Prediction of yielding

**Tresca**

$$\tau_e = \sigma_1 - \sigma_3 \leq \sigma_Y$$

(independent of $\mu_\sigma$)

$$\boxed{\frac{\sigma_1 - \sigma_3}{\sigma_Y} \leq 1}$$

**von Mises**

$$\sigma_2 = \mu_\sigma \frac{\sigma_1 - \sigma_3}{2} + \frac{\sigma_1 + \sigma_3}{2}$$

...substituting to HMH

$$\sigma_e = \underbrace{(\sigma_1 - \sigma_3)}_{\tau_e} \frac{\sqrt{3 + \mu_\sigma^2}}{2} \leq \sigma_Y$$

where

$$\frac{\sigma_1 - \sigma_3}{\sigma_Y} = \frac{\sigma_t}{\sigma_Y} = \frac{pr}{s\sigma_Y}$$

# Prediction of yielding

## Tresca

$$\tau_e = \sigma_1 - \sigma_3 \leq \sigma_Y$$

(independent of $\mu_\sigma$)

$$\boxed{\frac{\sigma_1 - \sigma_3}{\sigma_Y} \leq 1}$$

where

$$\frac{\sigma_1 - \sigma_3}{\sigma_Y} = \frac{\sigma_t}{\sigma_Y} = \frac{pr}{s\sigma_Y}$$

## von Mises

$$\sigma_2 = \mu_\sigma \frac{\sigma_1 - \sigma_3}{2} + \frac{\sigma_1 + \sigma_3}{2}$$

...substituting to HMH

$$\sigma_e = \underbrace{(\sigma_1 - \sigma_3)}_{\tau_e} \frac{\sqrt{3 + \mu_\sigma^2}}{2} \leq \sigma_Y$$

(remark: $\tau_e \leq 1.16\,\sigma_e$)

# Prediction of yielding

<u>Tresca</u>

$$\tau_e = \sigma_1 - \sigma_3 \leq \sigma_Y$$

(independent of $\mu_\sigma$)

$$\boxed{\frac{\sigma_1 - \sigma_3}{\sigma_Y} \leq 1}$$

where

$$\frac{\sigma_1 - \sigma_3}{\sigma_Y} = \frac{\sigma_t}{\sigma_Y} = \frac{pr}{s\sigma_Y}$$

<u>von Mises</u>

$$\sigma_2 = \mu_\sigma \frac{\sigma_1 - \sigma_3}{2} + \frac{\sigma_1 + \sigma_3}{2}$$

…substituting to HMH

$$\sigma_e = \underbrace{(\sigma_1 - \sigma_3)}_{\tau_e} \frac{\sqrt{3 + \mu_\sigma^2}}{2} \leq \sigma_Y$$

(remark: $\tau_e \leq 1.16\,\sigma_e$)

$$\boxed{\frac{\sigma_1 - \sigma_3}{\sigma_Y} \leq \frac{2}{\sqrt{3 + \mu_\sigma^2}}}$$

# Prediction of yielding

Tresca

$$\tau_e = \sigma_1 - \sigma_3 \le \sigma_Y$$

(independent of $\mu_\sigma$)

$$\boxed{\frac{\sigma_1 - \sigma_3}{\sigma_Y} \le 1}$$

where

$$\frac{\sigma_1 - \sigma_3}{\sigma_Y} = \frac{\sigma_t}{\sigma_Y} = \frac{pr}{s\sigma_Y}$$

von Mises

$$\sigma_2 = \mu_\sigma \frac{\sigma_1 - \sigma_3}{2} + \frac{\sigma_1 + \sigma_3}{2}$$

...substituting to HMH

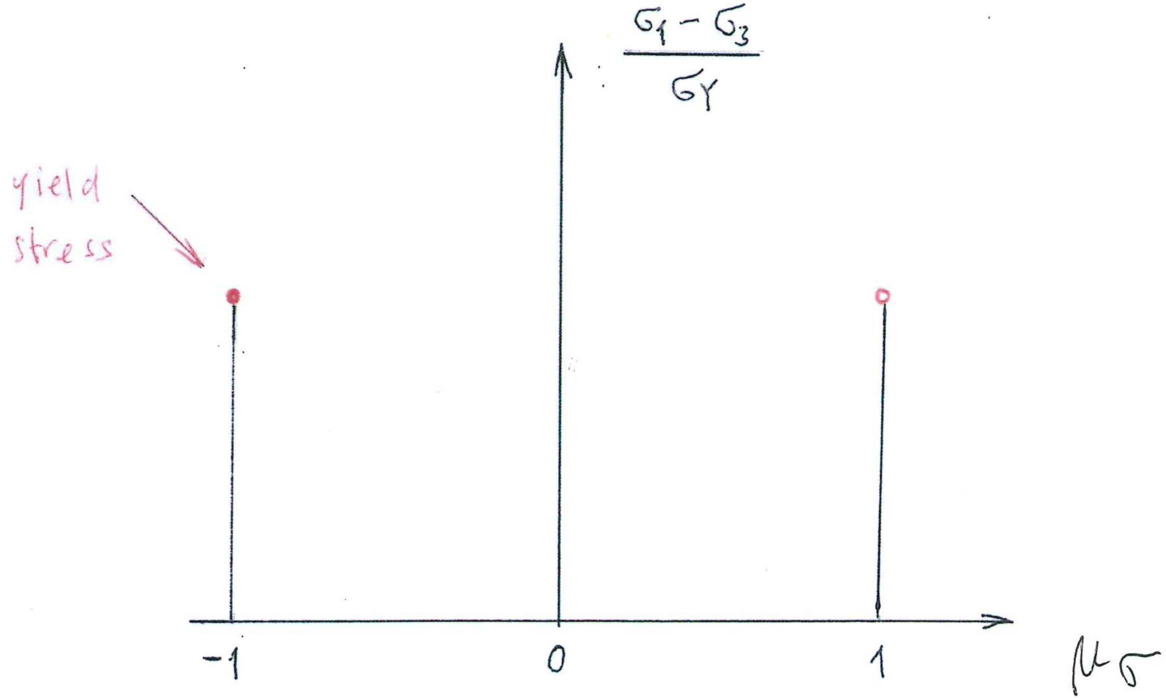
$$\sigma_e = \underbrace{(\sigma_1 - \sigma_3)}_{\tau_e} \frac{\sqrt{3 + \mu_\sigma^2}}{2} \le \sigma_Y$$

(remark: $\tau_e \le 1.16\,\sigma_e$)

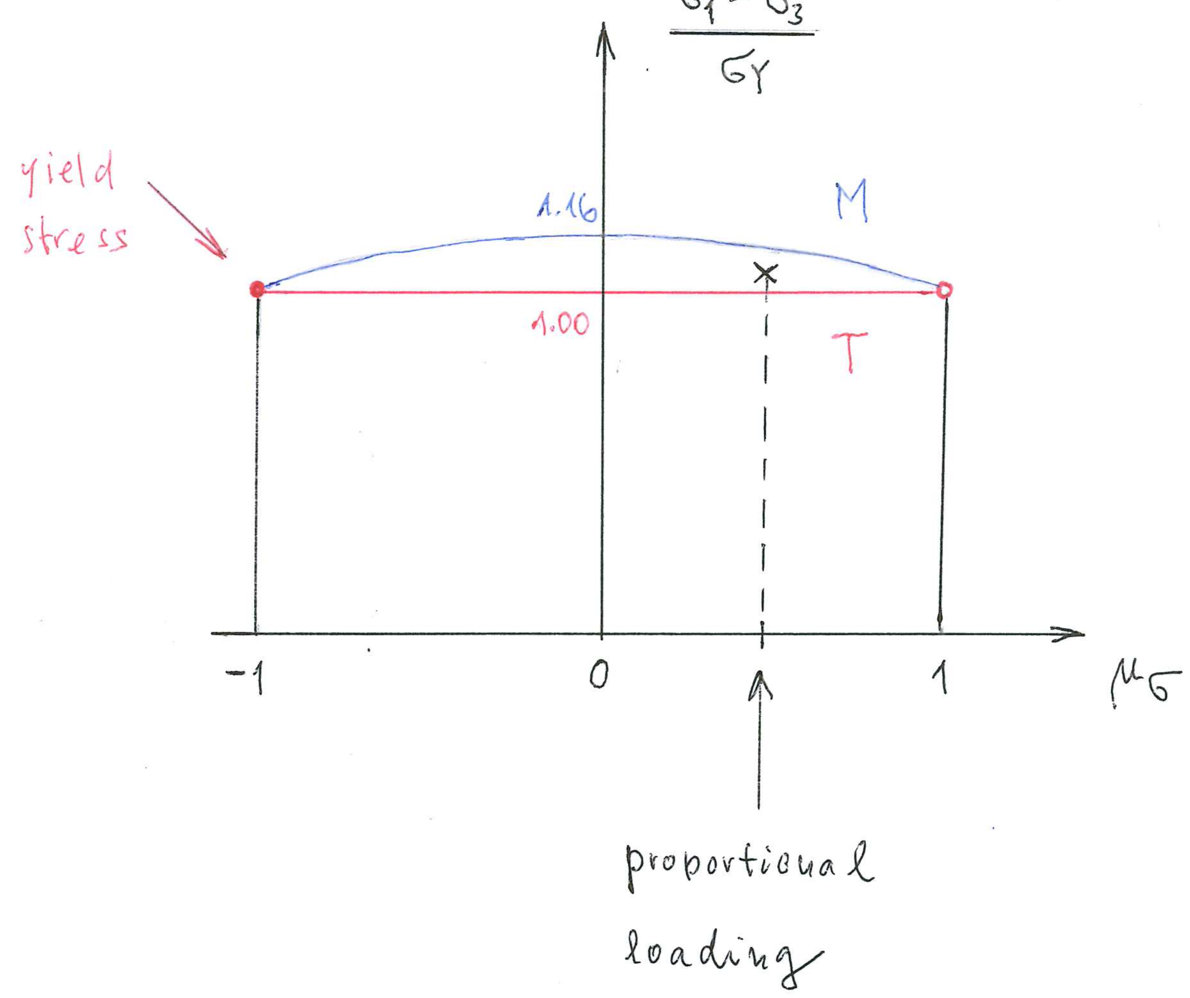
$$\boxed{\frac{\sigma_1 - \sigma_3}{\sigma_Y} \le \frac{2}{\sqrt{3 + \mu_\sigma^2}}}$$

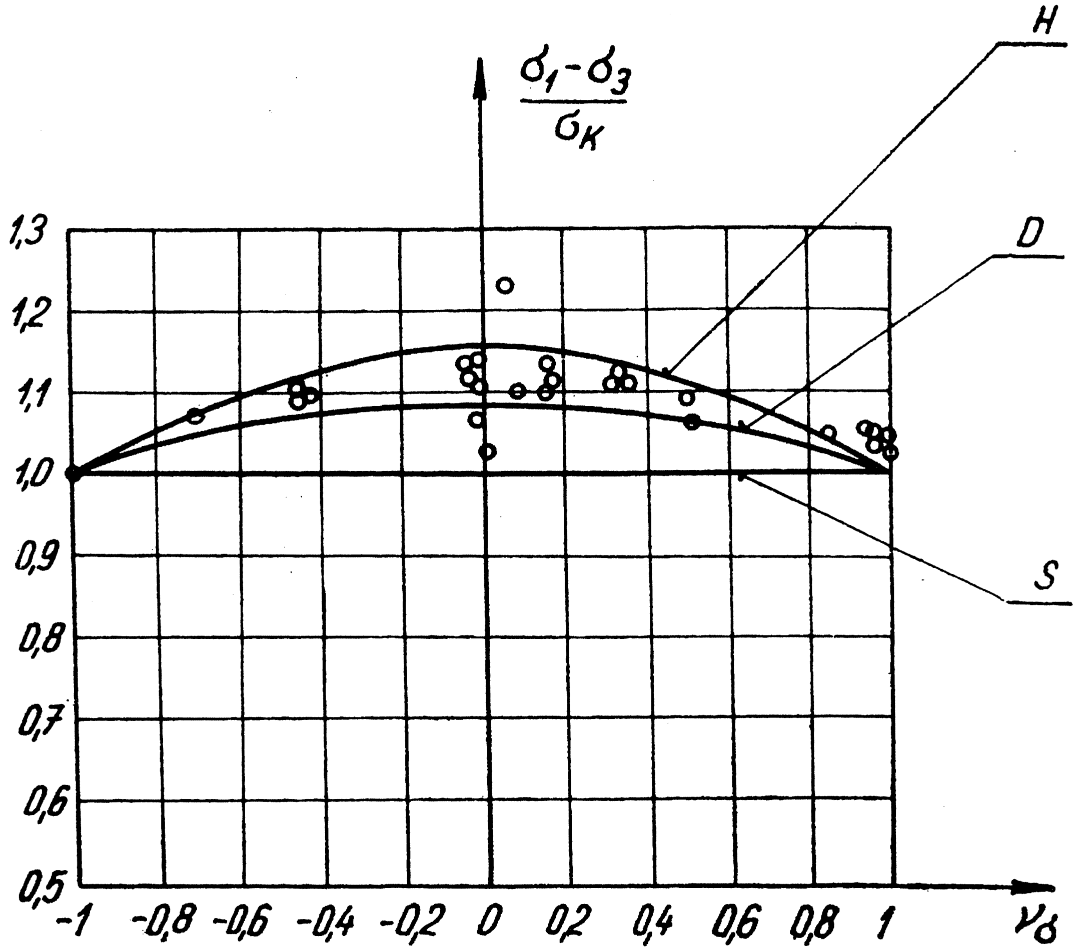
Property of Lode-Nádai setup: Load is controlled by $p$, yield limit by $\mu_\sigma$.

# Tresca versus Mises



# Tresca versus Mises






Průběhy hodnoty $\frac{\sigma_1 - \sigma_3}{\sigma_K}$ podle různých podmínek plasticity a experimentální hodnoty Nádaie a Lodea

# Taylor & Quinney (1931)

Tension-torsion of a thin wall tube

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma & \tau \\ \tau & 0 \end{bmatrix}$$

# Taylor & Quinney (1931)

Tension-torsion of a thin wall tube

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma & \tau \\ \tau & 0 \end{bmatrix}$$

Characteristic equation

$$\det|\boldsymbol{\sigma} - \lambda\mathbf{I}| = \lambda^2 - \sigma\lambda - \tau^2 = 0$$

# Taylor & Quinney (1931)

Tension-torsion of a thin wall tube

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma & \tau \\ \tau & 0 \end{bmatrix}$$

Characteristic equation

$$\det|\boldsymbol{\sigma} - \lambda \mathbf{I}| = \lambda^2 - \sigma\lambda - \tau^2 = 0$$

Principal stresses

$$\sigma_{1,3} = \frac{\sigma \pm \sqrt{\sigma^2 + 4\tau^2}}{2}$$

# Taylor & Quinney (1931)

Tension-torsion of a thin wall tube

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma & \tau \\ \tau & 0 \end{bmatrix}$$

Characteristic equation

$$\det |\boldsymbol{\sigma} - \lambda \mathbf{I}| = \lambda^2 - \sigma\lambda - \tau^2 = 0$$

Principal stresses

$$\sigma_{1,3} = \frac{\sigma \pm \sqrt{\sigma^2 + 4\tau^2}}{2}$$

Tresca

$$\boxed{\tau_e = \sqrt{\sigma^2 + 4\tau^2} \leq \sigma_Y}$$

# Taylor & Quinney (1931)

Tension-torsion of a thin wall tube

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma & \tau \\ \tau & 0 \end{bmatrix}$$

Characteristic equation

$$\det|\boldsymbol{\sigma} - \lambda \mathbf{I}| = \lambda^2 - \sigma\lambda - \tau^2 = 0$$

Principal stresses

$$\sigma_{1,3} = \frac{\sigma \pm \sqrt{\sigma^2 + 4\tau^2}}{2}$$

Tresca

$$\boxed{\tau_e = \sqrt{\sigma^2 + 4\tau^2} \leq \sigma_Y}$$

von Mises

$$\boxed{\sigma_e = \sqrt{\sigma^2 + 3\tau^2} \leq \sigma_Y}$$

# Taylor & Quinney (1931)

Tension-torsion of a thin wall tube

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma & \tau \\ \tau & 0 \end{bmatrix}$$

Characteristic equation

$$\det|\boldsymbol{\sigma} - \lambda \mathbf{I}| = \lambda^2 - \sigma\lambda - \tau^2 = 0$$

Principal stresses

$$\sigma_{1,3} = \frac{\sigma \pm \sqrt{\sigma^2 + 4\tau^2}}{2}$$

Tresca

$$\boxed{\tau_e = \sqrt{\sigma^2 + 4\tau^2} \leq \sigma_Y}$$

von Mises

$$\boxed{\sigma_e = \sqrt{\sigma^2 + 3\tau^2} \leq \sigma_Y}$$

Chart: ellipse equations

# Taylor & Quinney (1931)

Tension-torsion of a thin wall tube

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma & \tau \\ \tau & 0 \end{bmatrix}$$

Characteristic equation

$$\det|\boldsymbol{\sigma} - \lambda\mathbf{I}| = \lambda^2 - \sigma\lambda - \tau^2 = 0$$

Principal stresses

$$\sigma_{1,3} = \frac{\sigma \pm \sqrt{\sigma^2 + 4\tau^2}}{2}$$

Tresca

$$\boxed{\tau_e = \sqrt{\sigma^2 + 4\tau^2} \leq \sigma_Y}$$

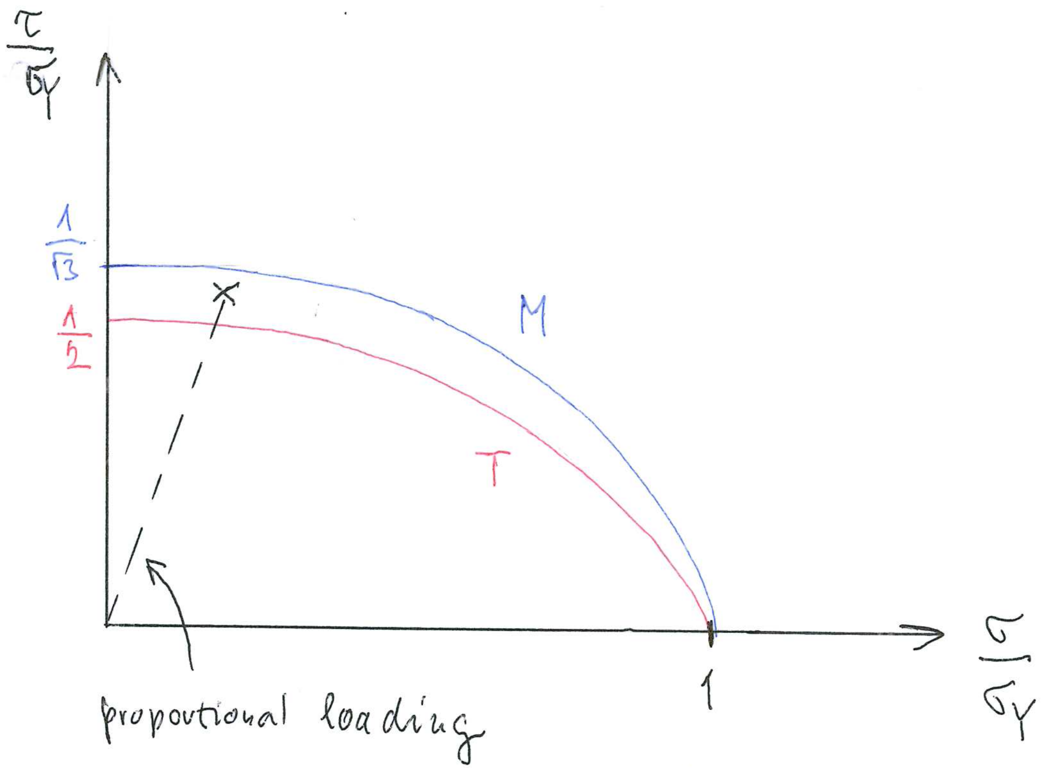
von Mises

$$\boxed{\sigma_e = \sqrt{\sigma^2 + 3\tau^2} \leq \sigma_Y}$$

Chart: ellipse equations

Remark: Only one twelfth of the deviatoric plane is covered.

# Tresca versus Mises



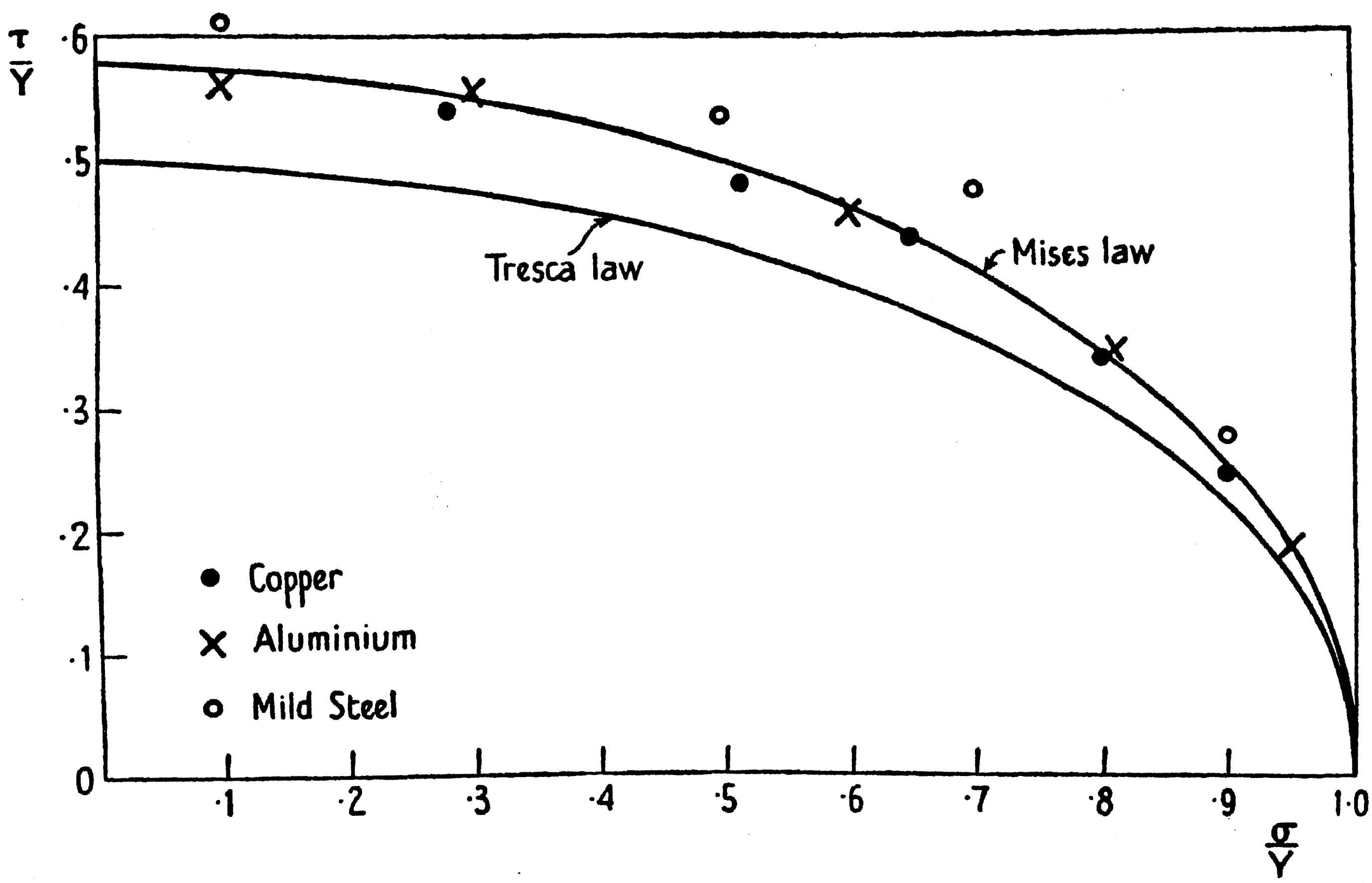



FIG. 4. Experimental results of Taylor and Quinney from combined torsion and tension tests, each metal being work-hardened to the same state for all tests. The Mises law is $\sigma^2 + 3\tau^2 = Y^2$, while the Tresca law is $\sigma^2 + 4\tau^2 = Y^2$, where $\sigma$ = tensile stress, $\tau$ = shear stress, $Y$ = tensile yield stress.