Institute of Thermomechanics ASCR

ENGINEERING PLASTICITY I

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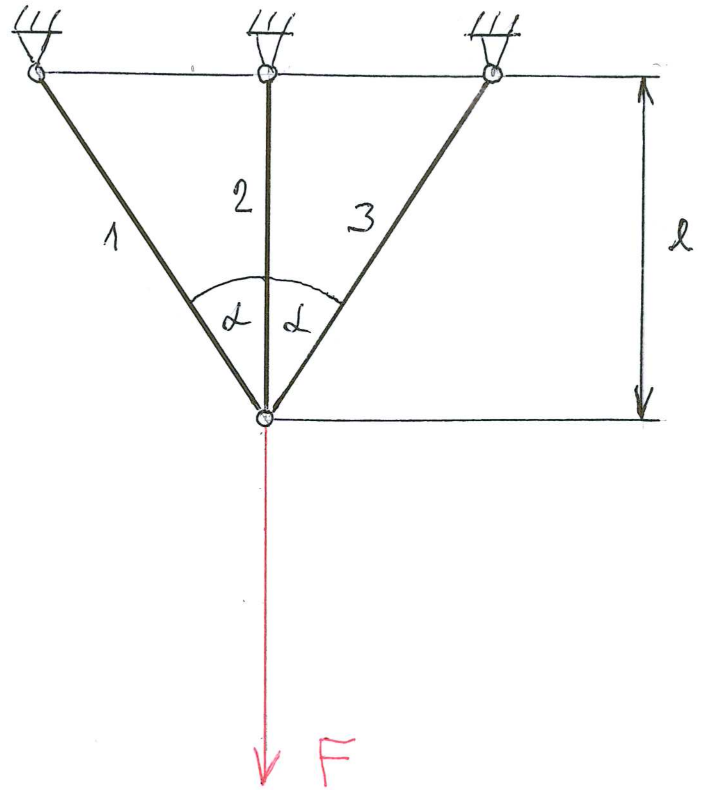
Contents

- Brittle and ductile fracture
- Elastic and plastic limits
- Static load analysis
- Residual stresses and shakedown analysis
- High and low cycle fatigue

Truss structure

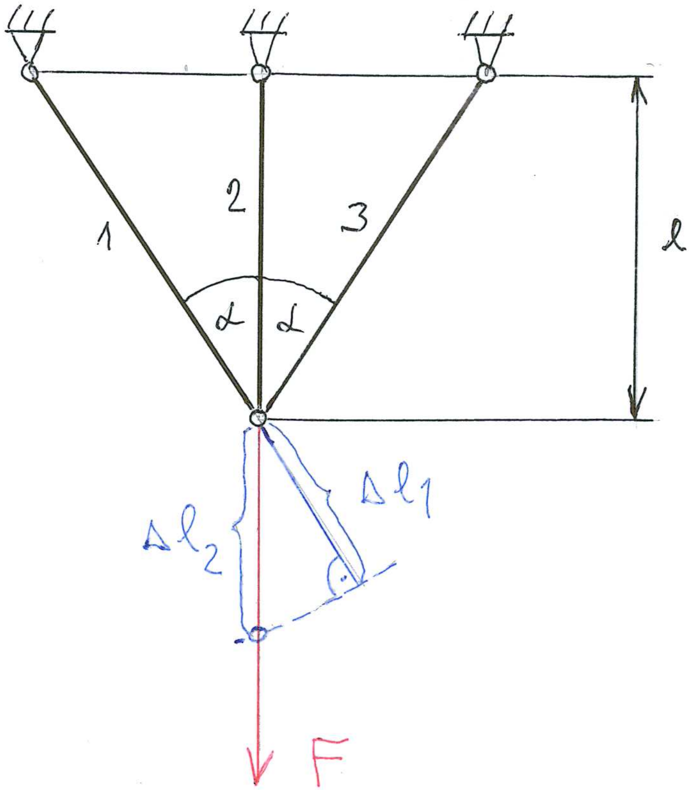


equilibrium

$$2N_1 \cos\alpha + N_2 = F$$

Load carrying capacity ?

Truss structure



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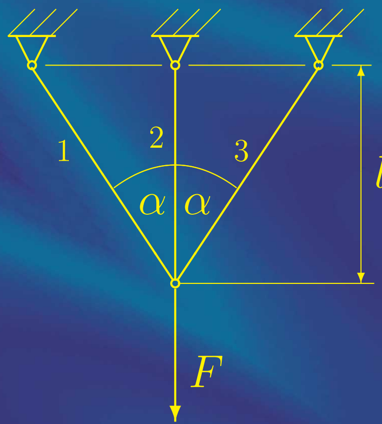
compatibility

$$\Delta l_2 \cos\alpha = \Delta l_1$$

Load carrying capacity ?

Elastic solution (1/2)



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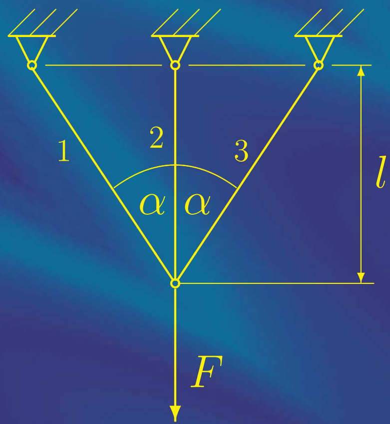

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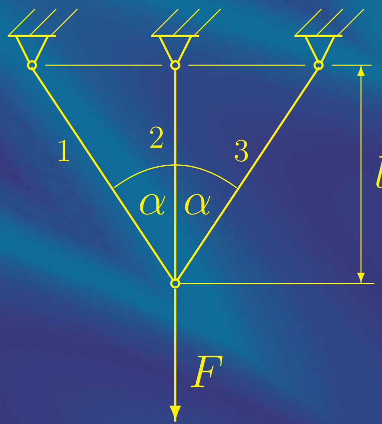
Hooke: $\Delta l_2 = \dfrac{N_2 l}{EA}\,, \quad \Delta l_1 = \dfrac{N_1 l}{EA}\dfrac{1}{\cos\alpha}$

Elastic solution (1/2)



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(Beltrami-Michell)

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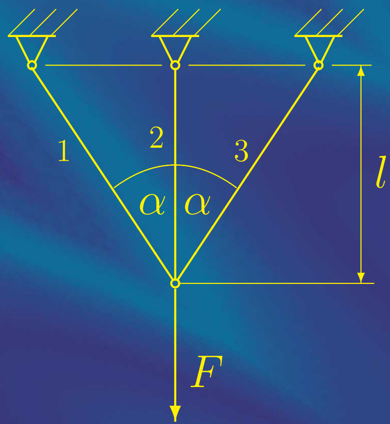

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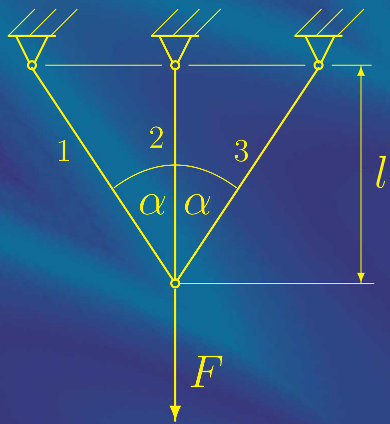
Question: Equilibrium versus BM equation?

Elastic solution (2/2)



$$N_1 = \frac{F\cos^2\alpha}{1+2\cos^3\alpha}$$

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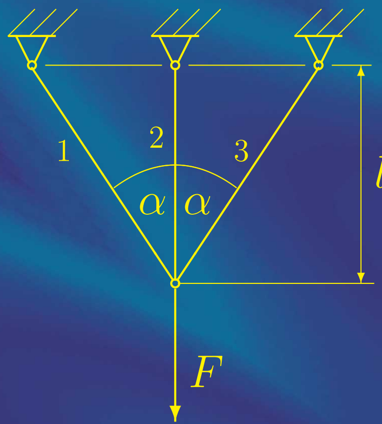

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Critical state

$$N_1 = N_3 < N_2 = N_{\text{crit}} \quad \Rightarrow \quad \boxed{F_{\text{crit}} = N_{\text{crit}}(1+2\cos^3\alpha)}$$

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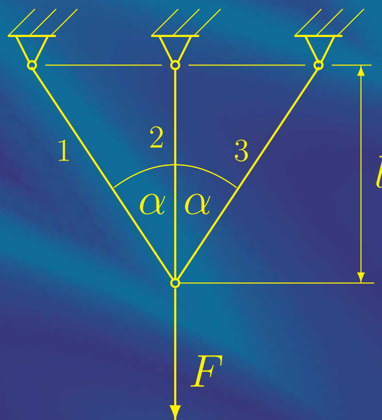

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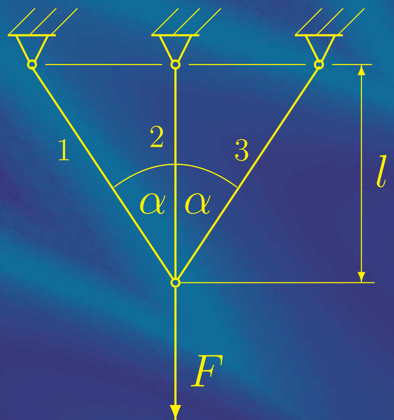
Remark: Independence on E, A, l. We speak about *a universal solution*.

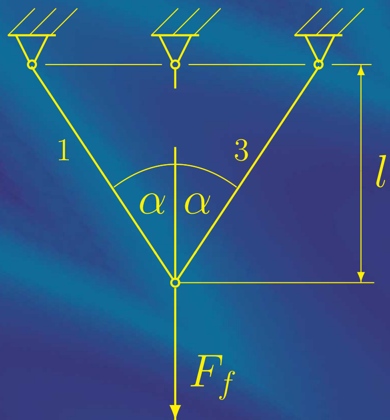
Brittle fracture



Let $N_{\mathsf{crit}} = N_f =$ force to rupture

The middle strut (2) ruptures at
$F_f = N_f(1 + 2\cos^3\alpha)$

Brittle fracture



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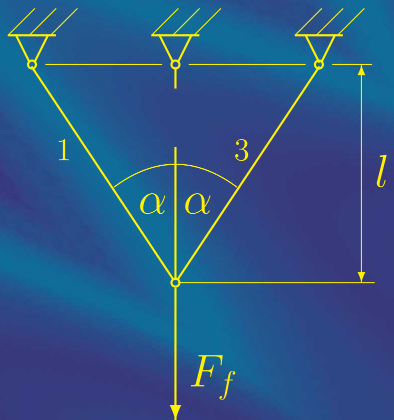

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Simplified equilibrium: $2N_1 \cos\alpha + 0 = F_f$

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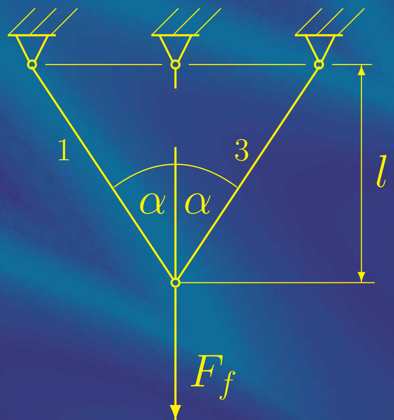

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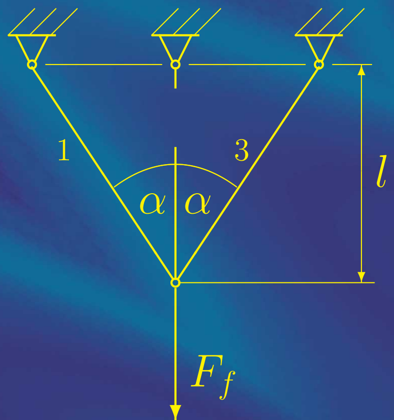
Brittle fracture



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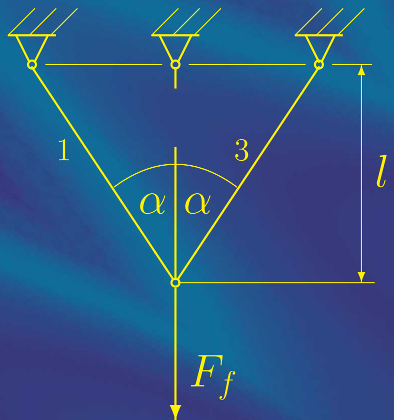

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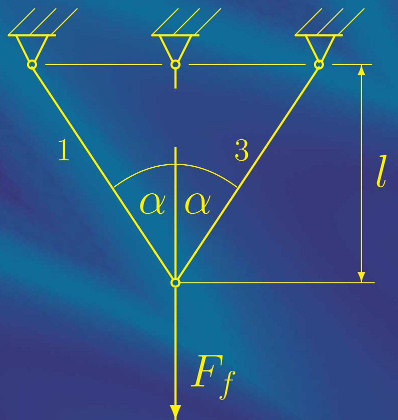

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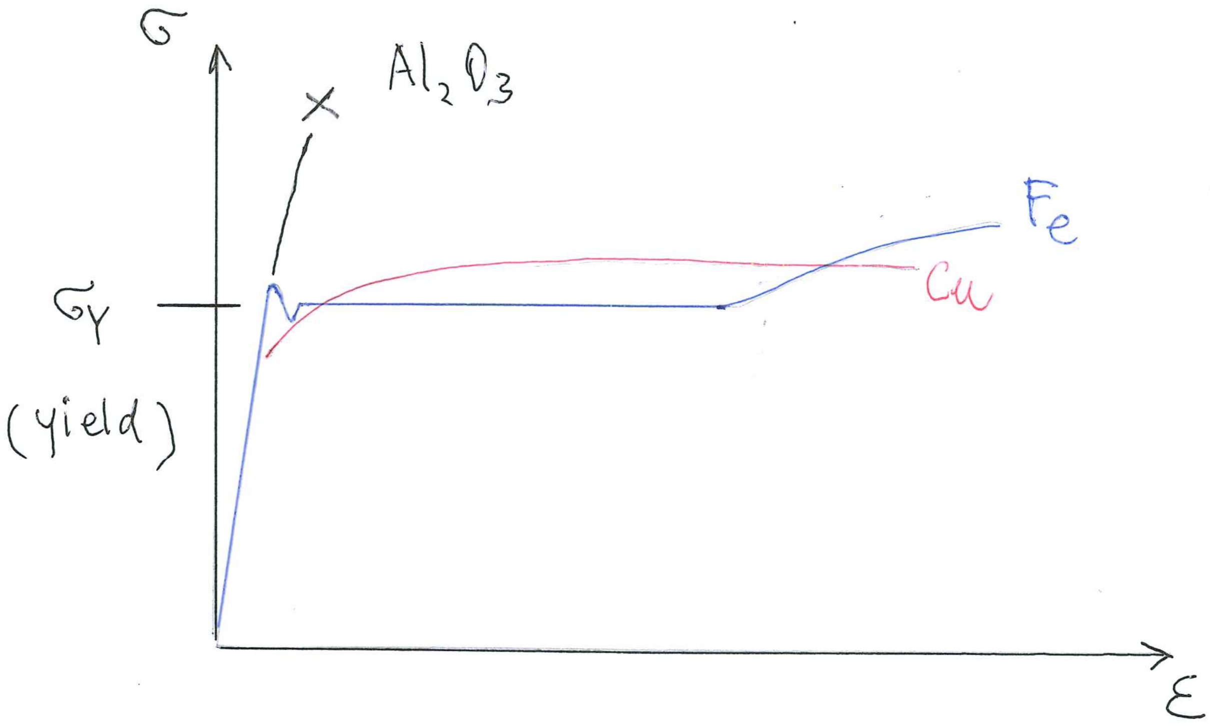
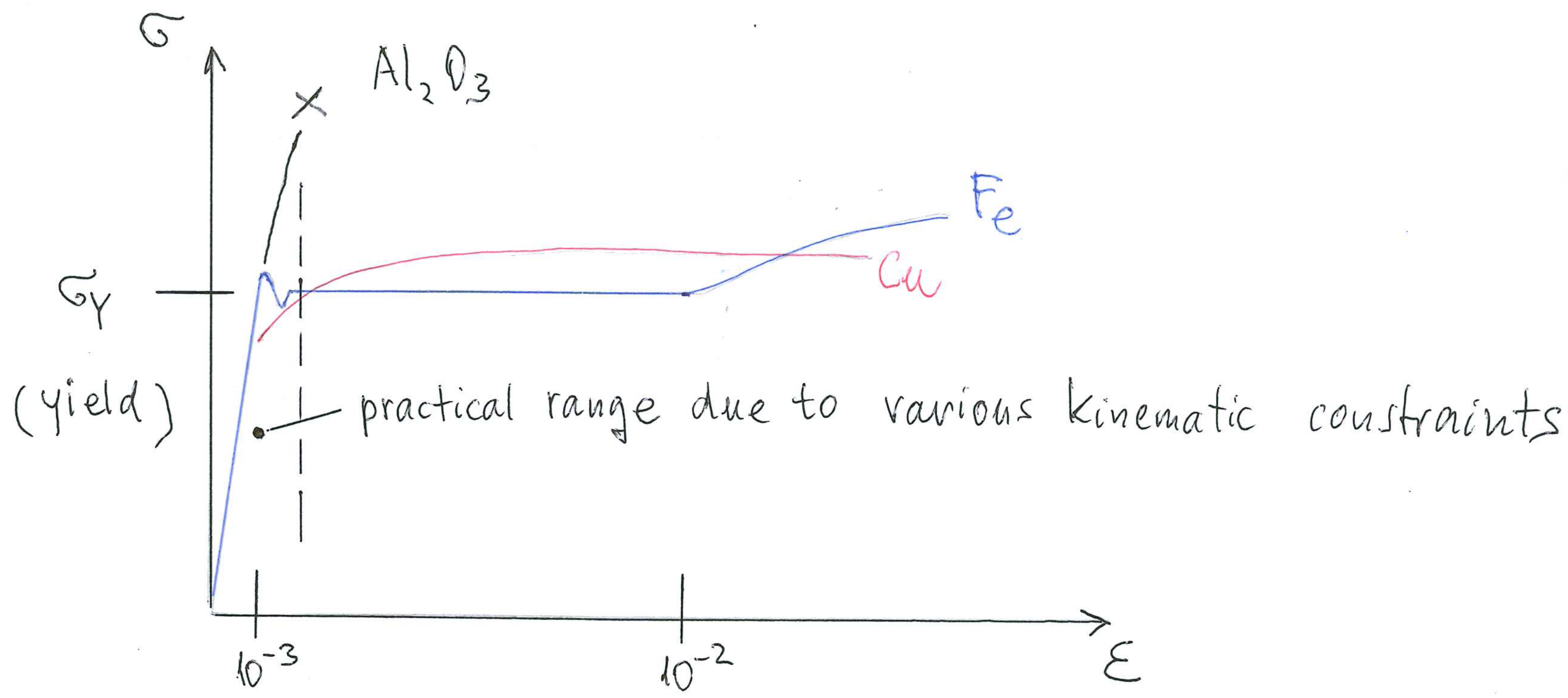
$$\boxed{F < F_f = N_f(1 + 2\cos^3\alpha)}$$

Remark: Ignoring (2) altogether a conservative condition follows as $F < 2N_{\text{crit}}\cos\alpha$

Typical stress-strain curves

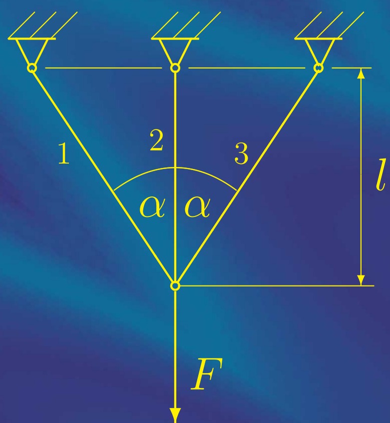


Typical stress-strain curves



Idealization $\sigma = \sigma_Y =$ const. is good and useful.

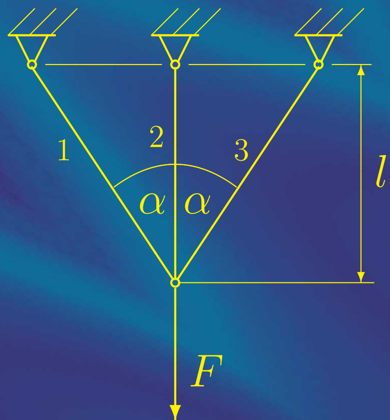
Ductile fracture



Let $N_{\text{crit}} = N_Y = A\sigma_Y$

The middle strut (2) yields at
$F_e = N_Y(1 + 2\cos^3\alpha)$

Ductile fracture



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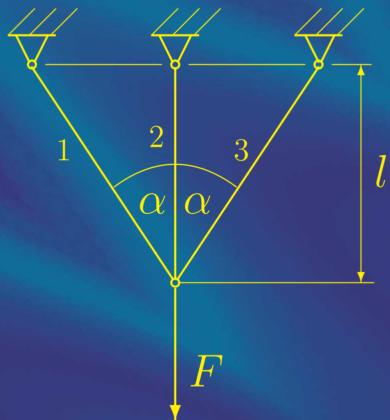

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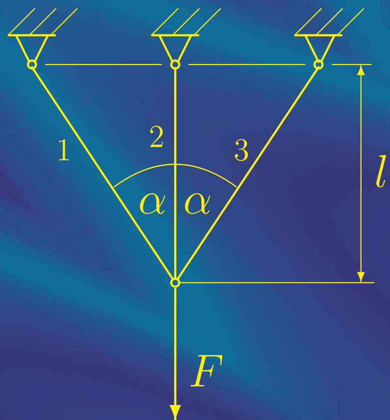

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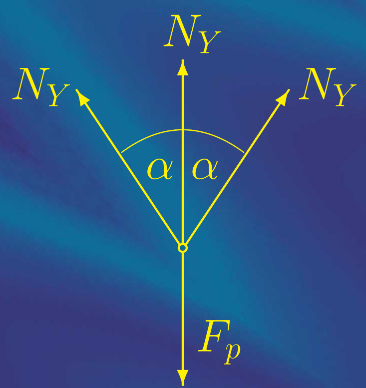
$$\boxed{F < F_p = N_Y(1 + 2\cos\alpha)}$$

Remark: No elastic solution needed!

Plastic limit analysis



Admissible force (stress) method

$$2N_Y \cos\alpha + N_Y = F_p$$

$$\boxed{F_p = N_Y(1 + 2\cos\alpha)}$$

Remark: Compatibility equation not needed!

Collapse load margin

Elastic-plastic load factor

$$\kappa = \frac{F_p}{F_e}$$

Collapse load margin

Elastic-plastic load factor

$$\kappa = \frac{F_p}{F_e}$$

Truss structure problem

$$\kappa = \frac{1 + 2\cos\alpha}{1 + 2\cos^3\alpha} = 1.2 \qquad (\text{for } \alpha = 30^\circ)$$

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Typical value for bending and torsion, $\kappa \simeq 1.5$, is used in many design codes.

Static load assessment

Load factor

$$\boxed{F < \frac{F_p}{k}, \quad k \simeq 1.5}$$

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Load factor

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Factor of safety

$$F < \frac{F_p}{k} = \frac{\kappa F_e}{k}$$

Static load assessment

Load factor

$$\boxed{F < \frac{F_p}{k}, \quad k \simeq 1.5}$$

Factor of safety

$$F < \frac{F_p}{k} = \frac{\kappa F_e}{k} = \frac{F_e}{k'}, \quad k' = \frac{k}{\kappa} \simeq 1$$

Static load assessment

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Allowable stress

$$\sigma_{\mathsf{max}} = f(F)$$

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Allowable stress

$$\sigma_{\mathsf{max}} = f(F) < f\left(\frac{F_e}{k'}\right)$$

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$$\sigma_{\mathsf{max}} = f(F) < f\left(\frac{F_e}{k'}\right) = \frac{1}{k'} f(F_e)$$

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$$\sigma_{\max} = f(F) < f\left(\frac{F_e}{k'}\right) = \frac{1}{k'}\underbrace{f(F_e)}_{\sigma_Y}$$

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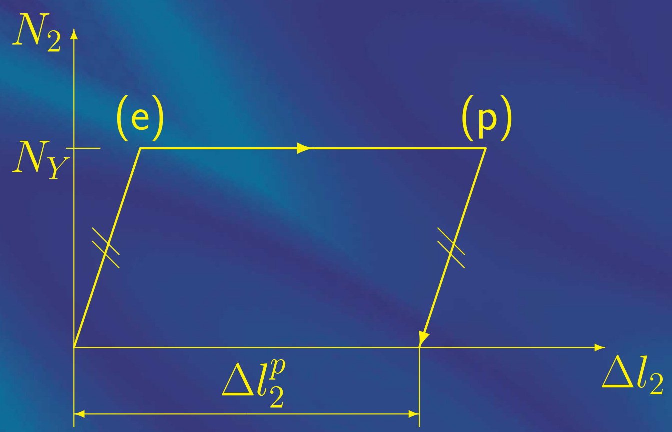
Elastic analysis only needs to be performed! Plasticity included via FoS.

Unloading

Both (1) and (3) load and unload elastically.



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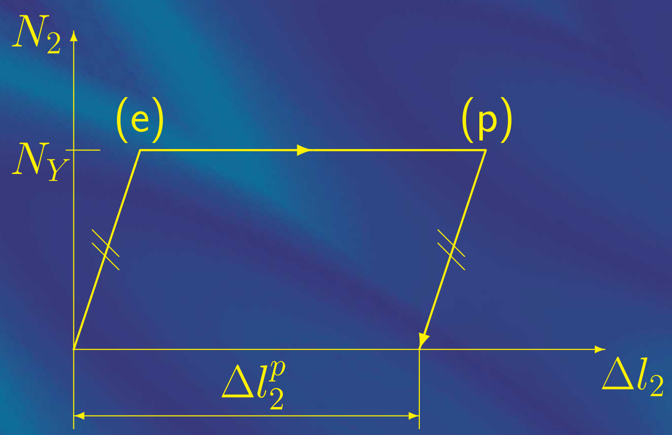
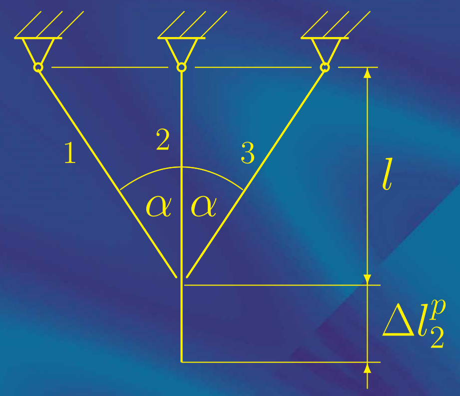


Unloaded and dismantled



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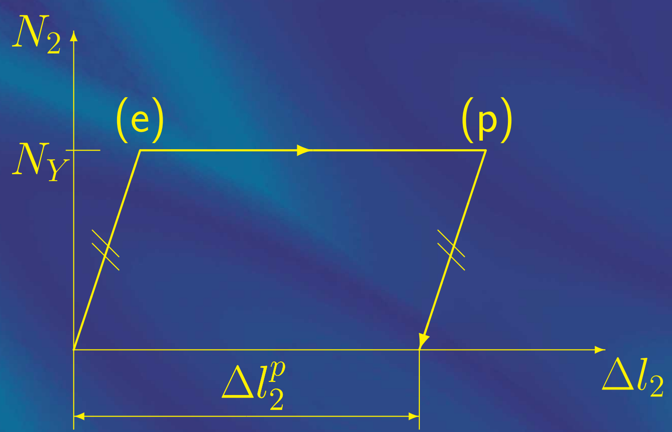

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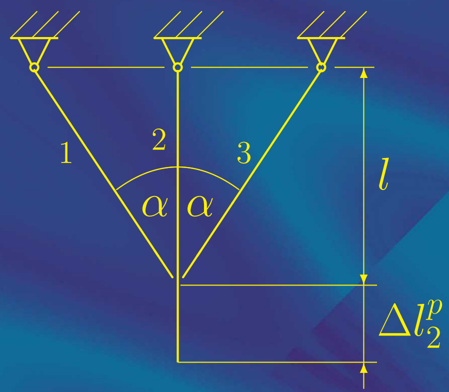

Residual stresses by solving a statically indeterminate problem or . . .

Residual stresses

Elastic solution

$$N_2^{\mathrm{el}}(F) = \frac{F}{1 + 2\cos^3\alpha} \qquad \text{(linear)}$$

Residual stresses

Elastic solution

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$$N_2(F) = \begin{cases} N_2^{\mathsf{el}}(F)\,, & F \le F_e \\ N_Y\,, & F > F_e \end{cases}$$

Residual stresses

Elastic solution

$$N_2^{\mathsf{el}}(F) = \frac{F}{1 + 2\cos^3\alpha} \quad \text{(linear)}$$

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Residual force

$N_2^{\mathsf{res}} = N_2(F) + N_2^{\mathsf{el}}(-F)$

Residual stresses

Elastic solution

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$$N_2(F) = \begin{cases} N_2^{\text{el}}(F)\,, & F \le F_e \\ N_Y\,, & F > F_e \end{cases}$$

Residual force

$N_2^{\text{res}} = N_2(F) + N_2^{\text{el}}(-F) = N_2(F) - N_2^{\text{el}}(F)\,,$ (as long as $N_2^{\text{res}} \ge -N_Y$)

Residual stresses

Elastic solution

$$N_2^{\mathrm{el}}(F) = \frac{F}{1 + 2\cos^3\alpha} \quad \text{(linear)}$$

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$$N_2^{\mathrm{res}} = N_2(F) + N_2^{\mathrm{el}}(-F) = N_2(F) - N_2^{\mathrm{el}}(F)\,, \qquad \text{(as long as } N_2^{\mathrm{res}} \ge -N_Y)$$

Limit state

$$N_2^{\mathrm{res}} = N_2(F_p) - N_2^{\mathrm{el}}(F_p)$$

Residual stresses

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Limit state

$N_2^{\text{res}} = N_2(F_p) - N_2^{\text{el}}(F_p) = N_2(F_p) - \kappa N_2^{\text{el}}(F_e)$

Residual stresses

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Residual stresses

Elastic solution

$$N_2^{\text{el}}(F) = \frac{F}{1 + 2\cos^3\alpha} \qquad \text{(linear)}$$

Define

$$N_2(F) = \begin{cases} N_2^{\text{el}}(F)\,, & F \le F_e \\ N_Y\,, & F > F_e \end{cases}$$

Residual force

$$N_2^{\text{res}} = N_2(F) + N_2^{\text{el}}(-F) = N_2(F) - N_2^{\text{el}}(F)\,, \qquad \text{(as long as } N_2^{\text{res}} \ge -N_Y\text{)}$$

Limit state

$$N_2^{\text{res}} = N_2(F_p) - N_2^{\text{el}}(F_p) = \underbrace{N_2(F_p)}_{N_Y} - \kappa\,\underbrace{N_2^{\text{el}}(F_e)}_{N_Y} = N_Y - \kappa N_Y$$

$$\boxed{N_2^{\text{res}} = N_Y(1-\kappa)}$$

For $\kappa \le 2$ shakedown occurs, i.e., the structure's response becomes fully elastic.

Life time prediction

High cycle fatigue

Low cycle fatigue

Life time prediction

High cycle fatigue

- Whole process is entirely elastic.
- Crack propagation is the culprit.
- Fatigue strength σ_c may be introduced so that $\sigma_{\text{max}} < \dfrac{\sigma_c}{k}$.

Low cycle fatigue

- Repeated plastic deformation in cycles.
- If $\kappa \leq 2$ do nothing.
- If $\kappa > 2$, compute $\Delta\epsilon^p$ and use a strain-life method (Coffin-Manson).