NUMERICAL INTEGRATION

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  explicit approach

  implicit approach

Voigt's notation (1/2)

Stress six-vector

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For symmetry, $\mathbf{R} = \mathbf{g}$.

Remark: UMAT output to enter the Newton-Raphson procedure.

The algebro-differential problem

Stress increment

$$\dot{\boldsymbol{\sigma}} = \mathbf{C}(\dot{\boldsymbol{\epsilon}} - \lambda \mathbf{R})$$

subject to

$$F(\boldsymbol{\sigma}) = 0$$

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- Plešek, J.: Numerická integrace konstitutivních vztahů. Inž. mech., **6**, No. 1, pp. 3–24, 1999.
- Plešek, J., Křístek, A.: Assessment of methods for locating the point of initial yield. Comp. Meth. Appl. Mech. Engrg., **141**, pp. 389–397, 1997.

Three commandments

1. Robustness is what an engineer really needs, putting the notion of method's accuracy on the back burner. Besides, 'accurate' does not necessarily mean 'effective.'

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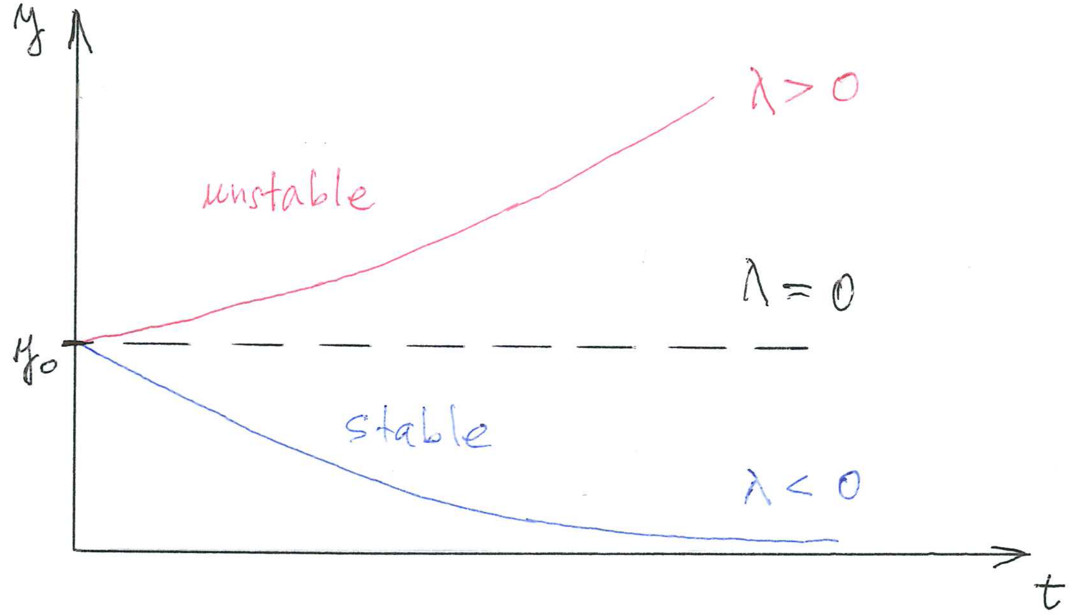
- Lambert, J.D.: *Numerical Methods for Ordinary Differential Systems.* Wiley, ISBN 0–471–92990–5, Chichester 1993.

Mathematical stability

ODE: $\dot{y} = \lambda y$, $\lambda \in R$ $\Rightarrow$ $\boxed{y(t) = y_0 e^{\lambda t}}$



Ljapunov stability: perturbation of initial conditions.

Explicit scheme

Euler forward

$$y_{n+1} = y_n + \dot{y}_n \Delta t$$

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Assuming math stability

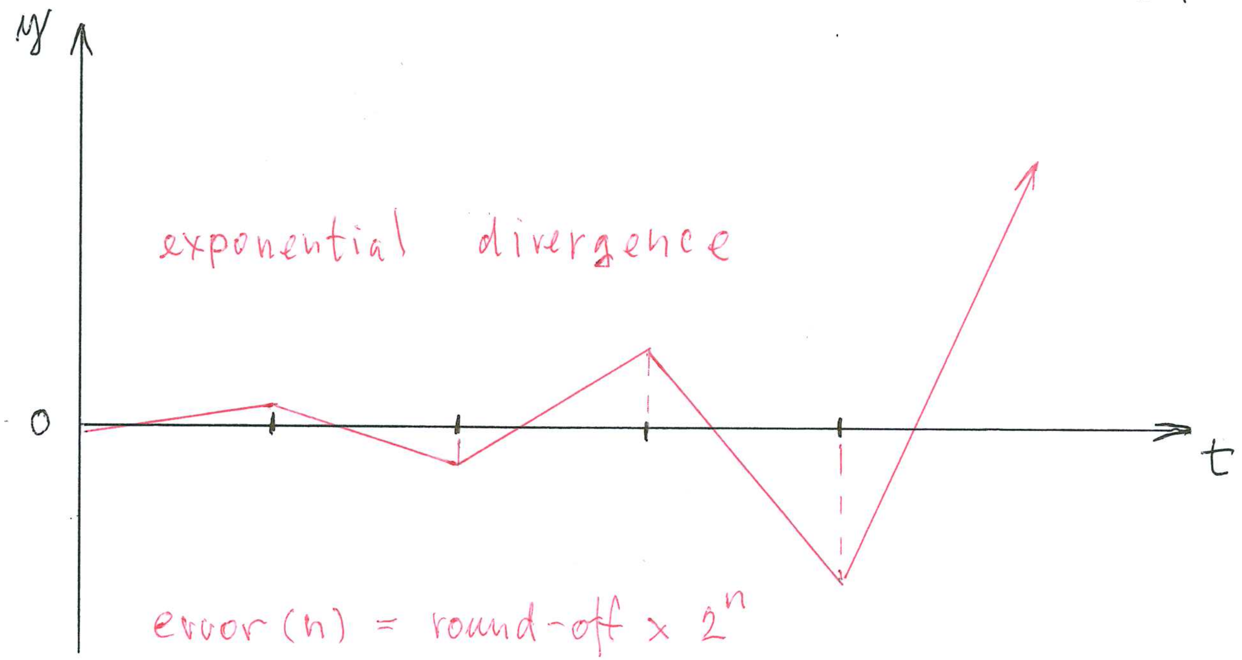
$$\boxed{\Delta t < \Delta t_{\text{crit}} = \frac{2}{|\lambda|}}$$

The Euler forward method is said to be conditionally stable.

Explicit scheme

Example: $\Delta t = 1.5\,\Delta t_{crit} = \dfrac{3}{|\lambda|}$

$$y_{n+1} = \left(1 + \lambda \frac{3}{|\lambda|}\right) y_n = -2\,y_n$$



for $n = 10$: $2^{10} = 1024$ $\Rightarrow$ for $n = 50$: $2^{50} \simeq 10^{15}$ (killing solution)

Remark: one unstable harmonic is sufficient to explode SODE.

Implicit scheme

Euler backward

$y_{n+1} = y_n + \dot{y}_{n+1}\Delta t = y_n + \lambda y_{n+1}\Delta t$

Recurrent formula

$$\boxed{y_{n+1} = \frac{y_n}{1 - \lambda\Delta t}}$$

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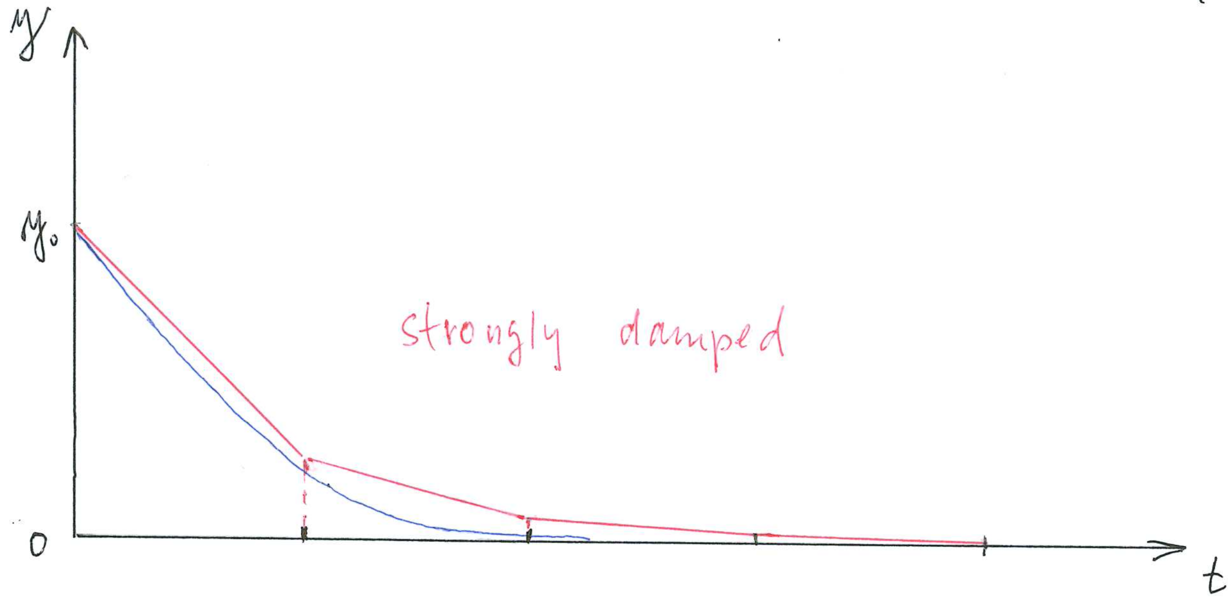
- Both methods have the same (first order or linear) accuracy, the only difference being their stability.

Implicit scheme

Example: $\lambda \Delta t = -3$ (as before)

$$y_{n+1} = \frac{y_n}{1-\lambda\Delta t} = \frac{1}{4} y_n$$



Remark: fast convergence toward static solution for $\Delta t \to \infty$.

Newmark method, DXDR method ...

A less known deficiency

Fixed point iteration

$$y_{n+1}^{(i+1)} = y_n + \lambda y_{n+1}^{(i)} \Delta t$$

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Convergence condition

$$\boxed{\Delta t < \frac{1}{|\lambda|} = \frac{1}{2} \Delta t_{\text{crit}}}$$

Remark: The critical time step is half the size of that of the Euler forward method!

Sets of ODE

Plasticity equations

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may, first, be linearized as $\mathcal{E}$rror $\rightarrow 0$ and, second, decoupled by the spectral decomposition to obtain

$$\mathbf{J} = \frac{\partial \mathbf{r}}{\partial \boldsymbol{\sigma}} \quad \text{and} \quad |\lambda|_{\mathsf{max}} = \max_i |\lambda_i|$$

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Remark: $|\lambda|_{\mathsf{max}}$ is seldom known beforehand, leaving us with experimentation.

Concluding remarks

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- One boast of implicit schemes is the built in early warning mechanism triggered by the divergence of the inner iteration loop. This is usually dealt with by halving the integration step. The integration is, thus, less efficient but able to run in a fully automated mode.