# THEORY OF PLASTIC FLOW I

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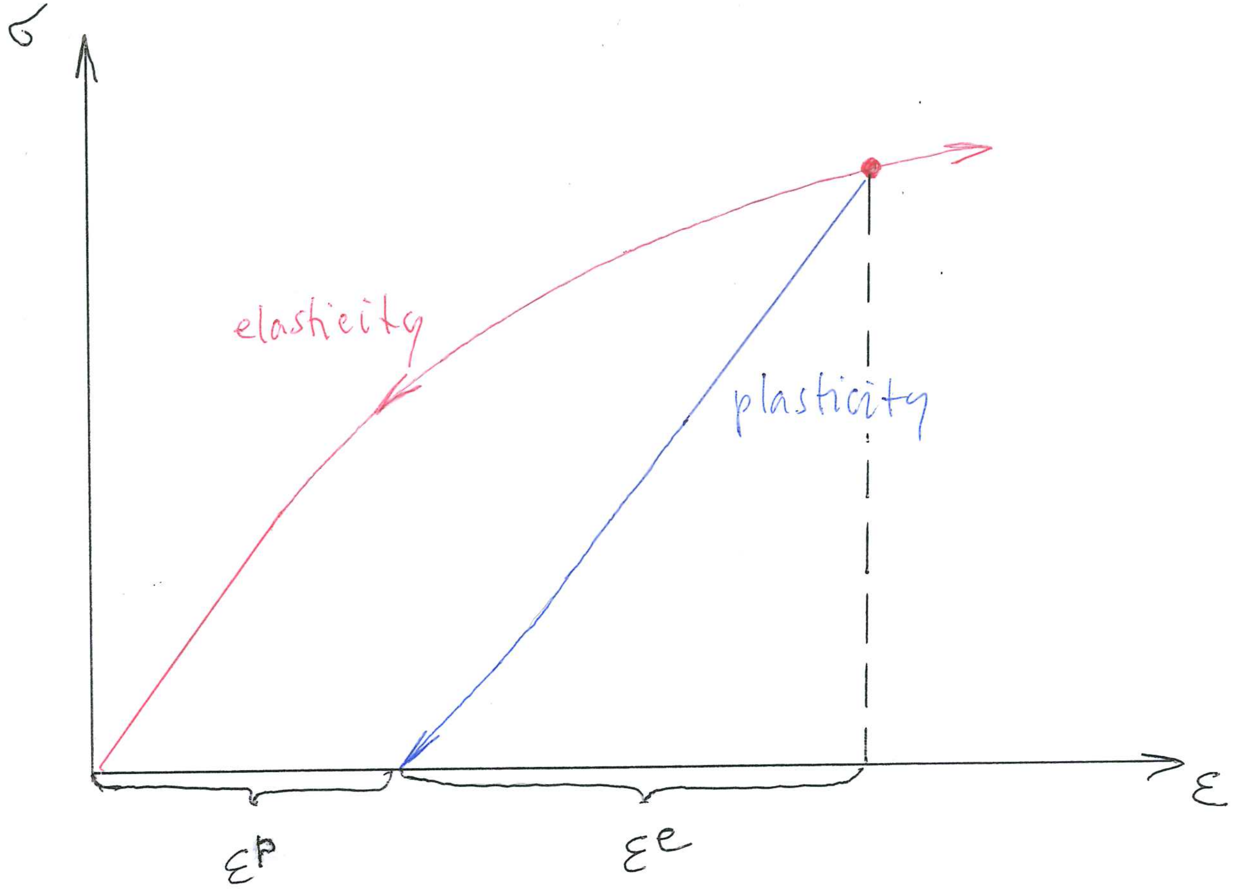
# Contents

# Paradigm of plasticity




(additive decomposition)

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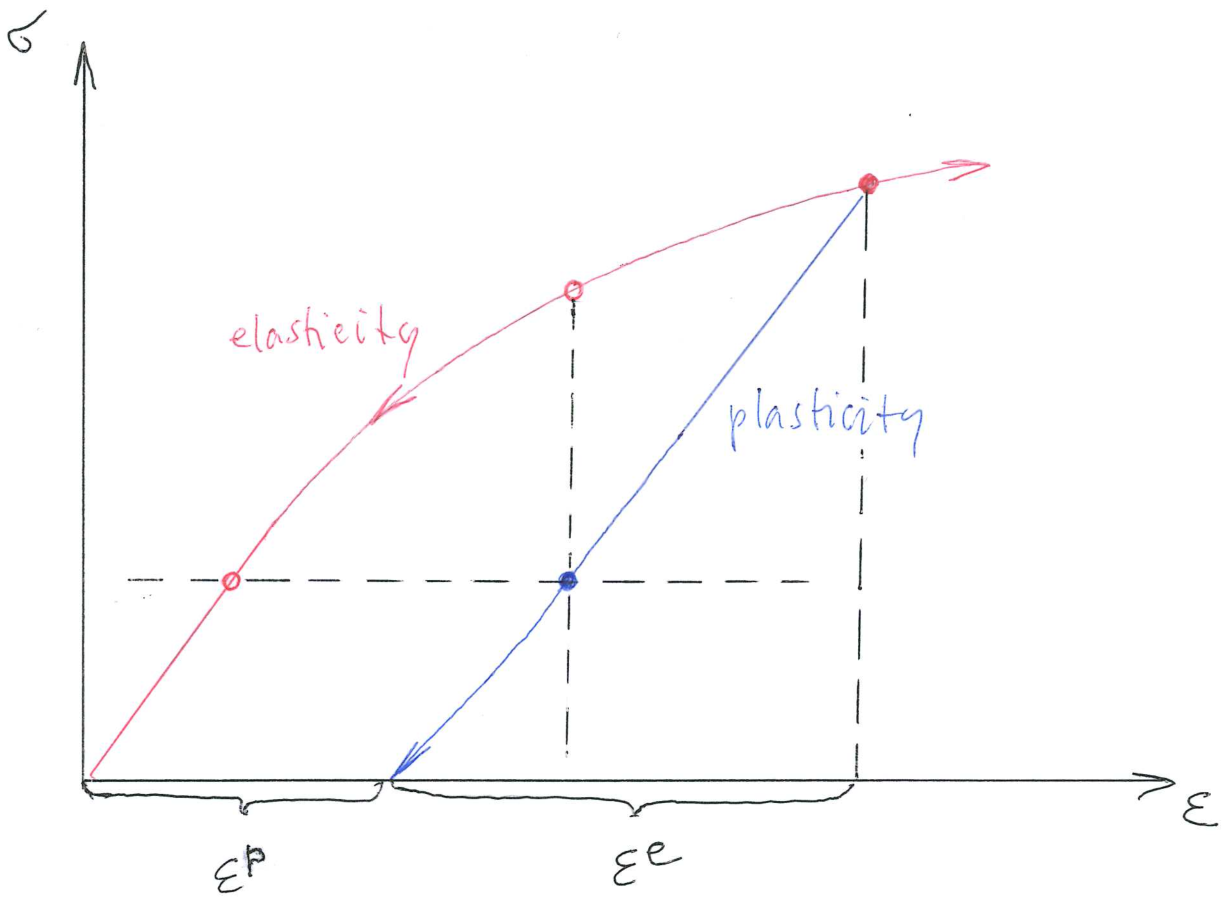


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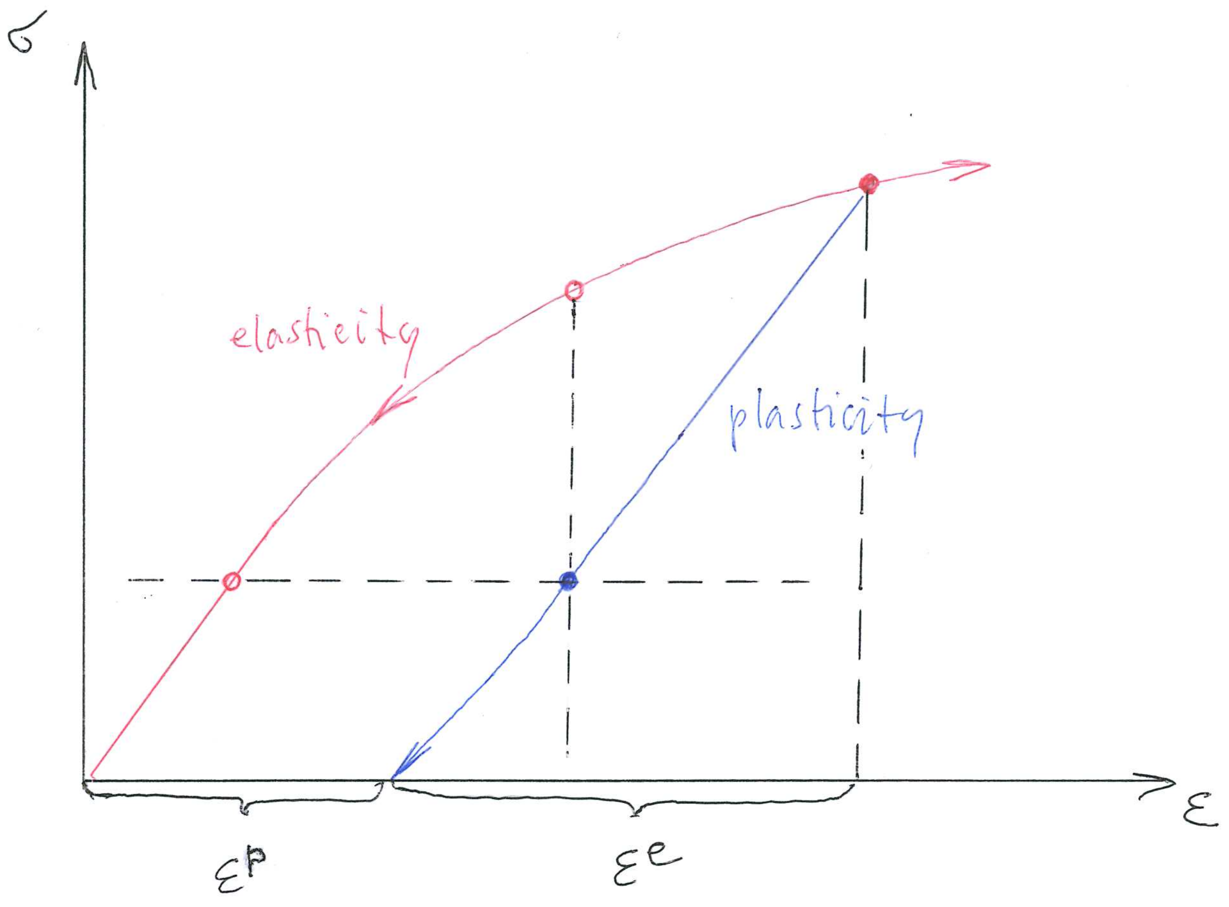


(additive decomposition)

## Function

$\varepsilon(t) \longmapsto \sigma(t)$

(elasticity)

## Functional

history: $0 \le \tau \le t$

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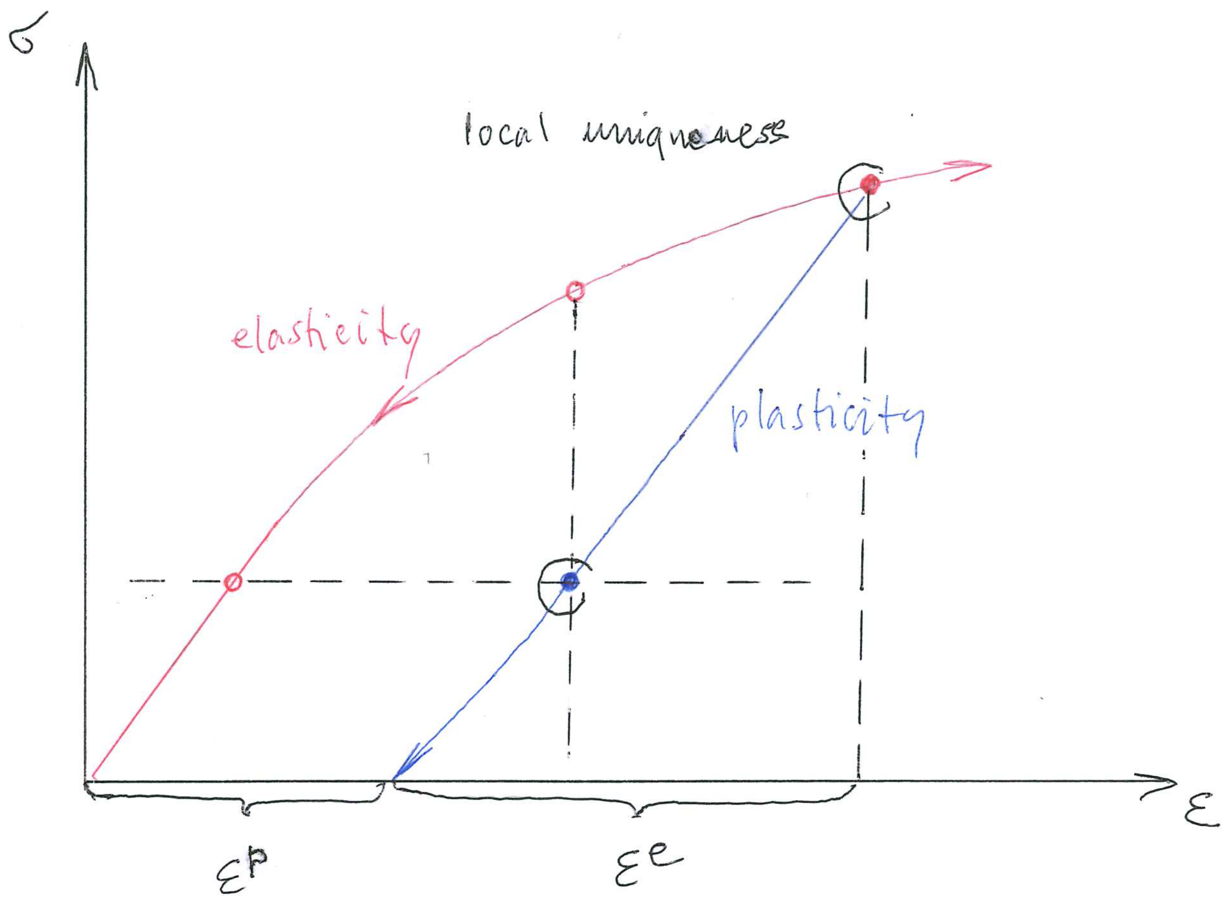


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Elastic unloading

if $\sigma\dot{\epsilon} < 0\,,$ we set $\lambda = 0$

# Hooke's law revisited

General formulation

$$\boxed{\sigma_{ij} = C_{ijkl}\epsilon_{kl}}$$

Symmetries

$$i \leftrightarrow j\,, \quad k \leftrightarrow l\,, \quad ij \longleftrightarrow kl$$

Positive definiteness

$$\forall \boldsymbol{\epsilon} \neq \mathbf{0}: \;\; \epsilon_{ij} C_{ijkl} \epsilon_{kl} > 0$$

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Isotropy

$$\mathbf{S} = 2G(\boldsymbol{\epsilon} - \epsilon_m\mathbf{I}) \qquad \text{and} \qquad \sigma_m = 3K\epsilon_m$$

where

$$G = \frac{E}{2(1+\nu)} \qquad \text{and} \qquad K = \frac{E}{3(1-2\nu)}$$

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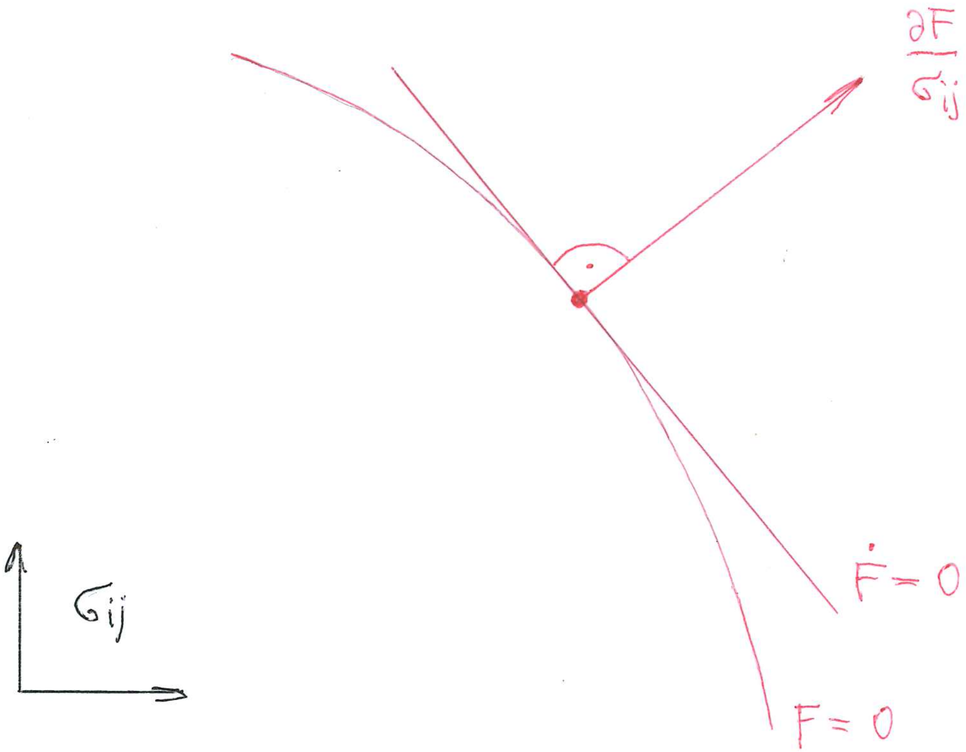
Hence

$$\boxed{\lambda = \frac{\dfrac{\partial F}{\partial \sigma_{ij}} C_{ijkl} \dot{\epsilon}_{kl}}{\dfrac{\partial F}{\partial \sigma_{pq}} C_{pqrs} R_{rs}} \geq 0}$$

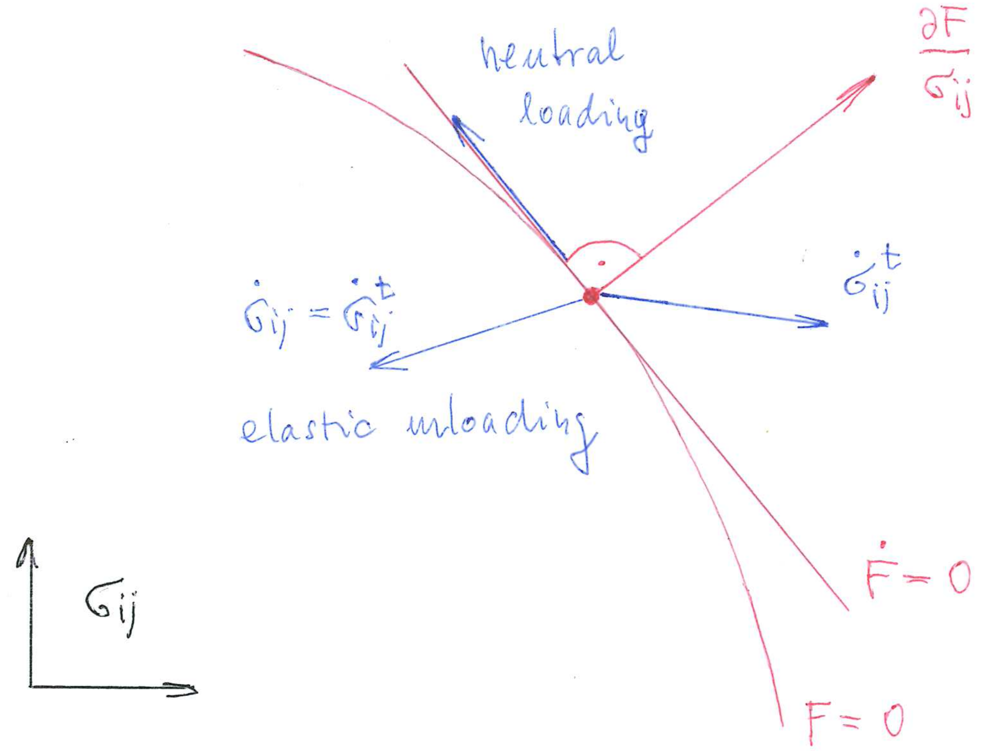
# Trial stress:



Trial stress: $\dot{\sigma}_{ij}^{t} = C_{ijkl}\,\dot{\varepsilon}_{kl}$

Plastic forcing

$$\lambda = \frac{\frac{\partial F}{\partial \sigma_{ij}} C_{ijkl}\,\dot{\varepsilon}_{kl}}{\text{const.}}$$



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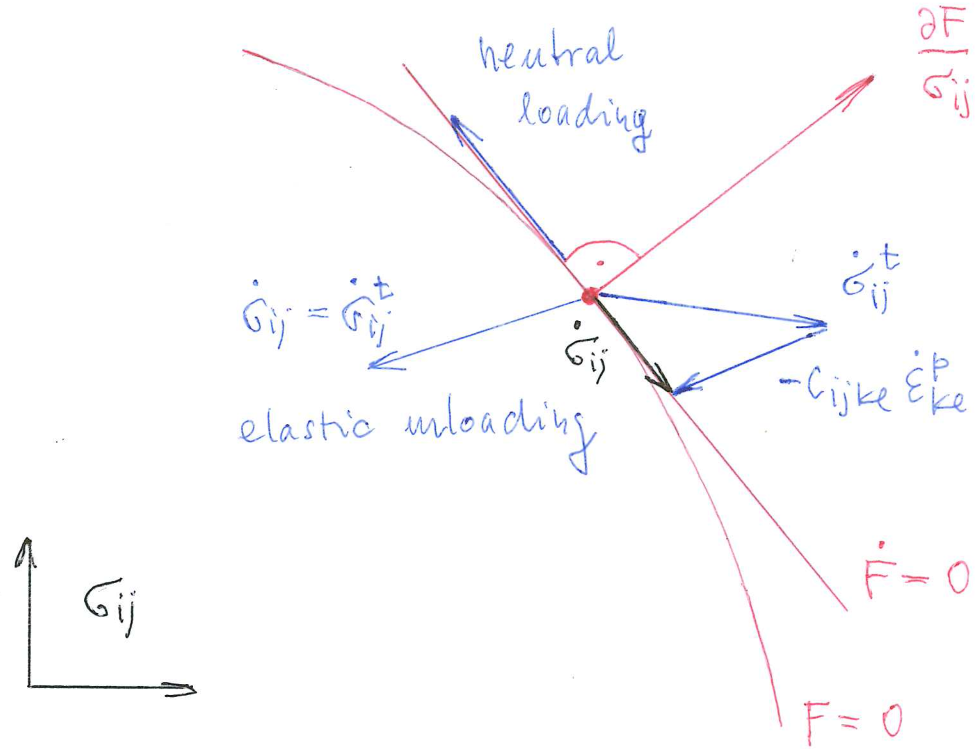


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Necessary condition

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Alternatively

$$C_{ijkl} R_{kl} = \frac{\partial F}{\partial \sigma_{ij}} \qquad \text{(interesting possibility)}$$

Note: AFR always meets the necessary condition as $\mathbf{C}$ is sym+def.

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Effective stress

$$\sigma_e = \sqrt{3J_2} = \sqrt{\frac{3}{2}S_{ij}S_{ij}}$$

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$$\dot{\sigma}_{ij} = \dot{\sigma}^t_{ij} - 2G\dot{\epsilon}^p_{ij}$$

Remark: Note that $S_{ij}\dot{\epsilon}_{ij}$ may be regarded as distortional power density.

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- We infer by $\dot{\boldsymbol{\epsilon}} = \mathbf{0} \;\Rightarrow\; \boldsymbol{\sigma} = \mathbf{0} \;\Rightarrow\; \boldsymbol{\epsilon}^e = \mathbf{0}$ that elastic strain is omitted, therefore this theory is only good for steady state flow.

# Von Mises (1913)

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$$\sqrt{\frac{3}{2}}\|\mathbf{S}\| \leq \sigma_Y$$

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- The flow rule $\dot{\epsilon}_{ij} = S_{ij}/2\mu$ is called today the Vénant-Lévy-Mises equation. Still, as we know already, it is incomplete.

# Prandtl-Reuss (1930)

- It was Reuss who, finally, included elastic strain via the additive decomposition so that only the plastic part would then be computed from VLM equation as

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- This, in fact, corresponds to $J_2$-theory with $\lambda_p = 3\lambda/2\sigma_e$ substituted. Either of the multipliers $\lambda$ or $\lambda_p$ can be determined from the consistency condition to the exact same result.