Institute of Thermomechanics ASCR

# ENGINEERING PLASTICITY III


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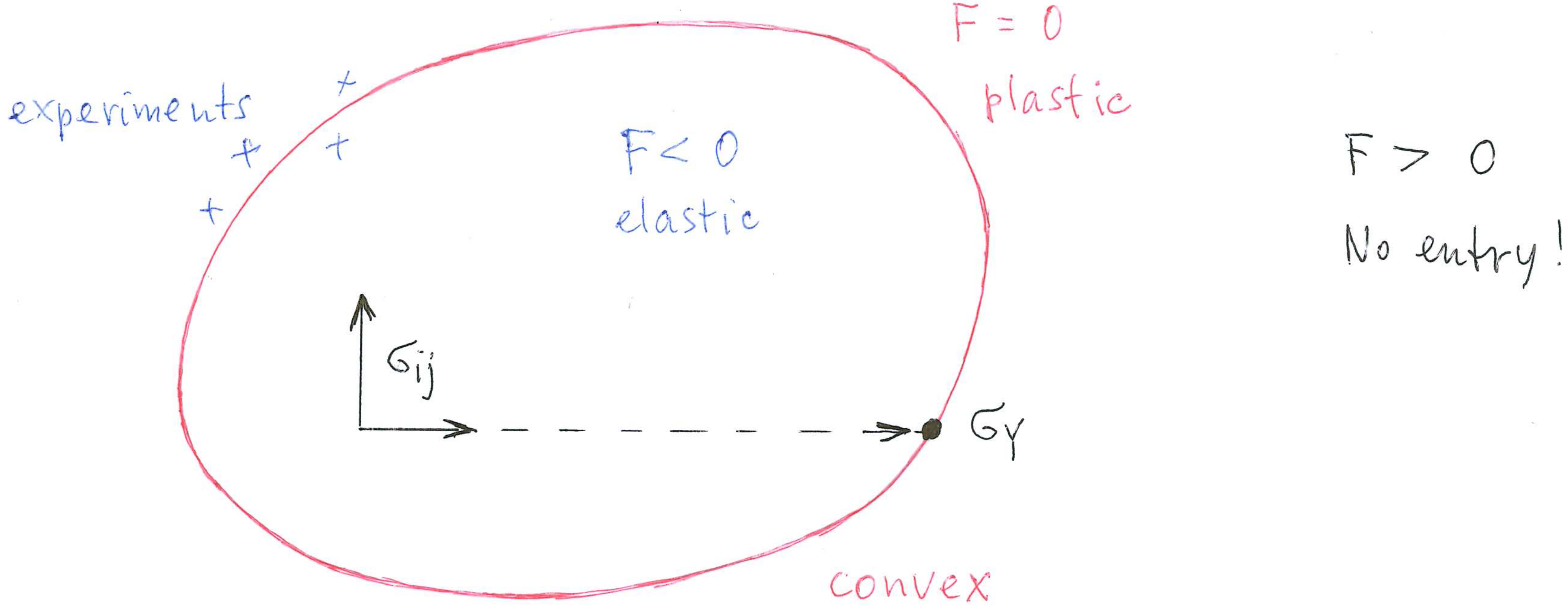
# Contents

# Yield function

$F(\sigma_{ij}) \leq 0$




Convexity does not follow from physics !!

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$$F(\sigma_{ij}) \equiv f(\sigma_{ij}) - \sigma_Y$$

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5. Validation

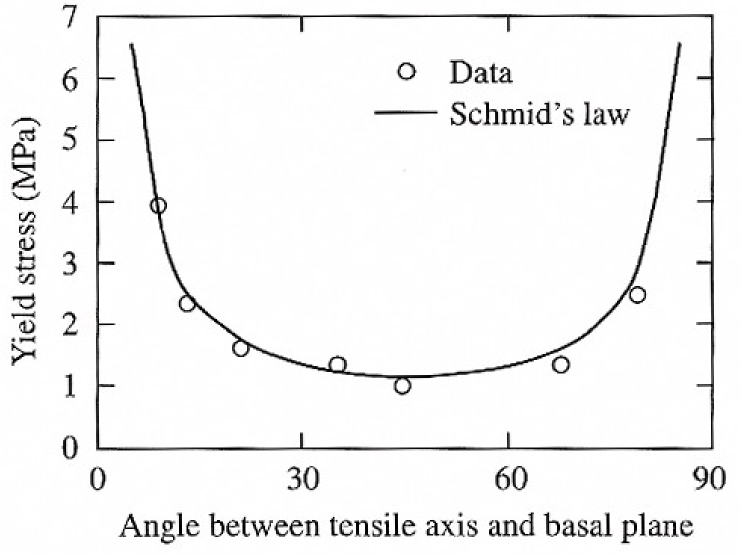
...Luders (1860), Schmid (1924)

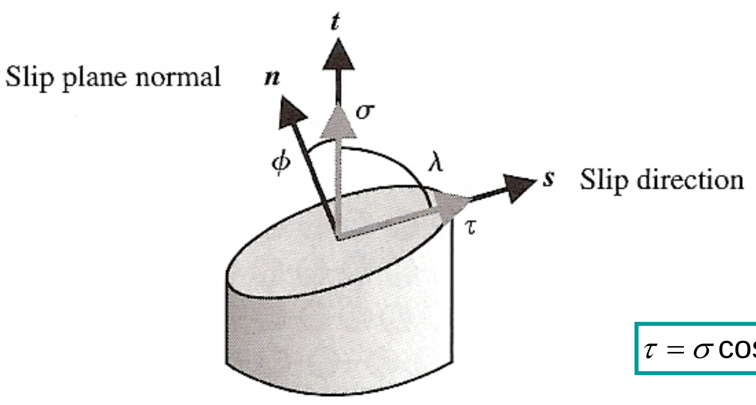
Remark: Predictions can (and should) be validated.

# Luders bands (1860)

# Single crystals – Schmid's law

| Metal | Structure | Slip systems | CRSS (MPa) |
|---|---|---|---|
| Cu | fcc | $\langle 1\bar{1}0\rangle\{111\}$ | 0.64 |
| Al | fcc | $\langle 1\bar{1}0\rangle\{111\}$ | 0.40 |
| Ni | fcc | $\langle 1\bar{1}0\rangle\{111\}$ | 5.8 |
| $\alpha$-Fe | bcc | $\langle 11\bar{1}\rangle\{101\}$, $\langle 11\bar{1}\rangle\{112\}$, $\langle 11\bar{1}\rangle\{123\}$ | 32 |
| Mo | bcc | $\langle 11\bar{1}\rangle\{101\}$ | 50 |
| Ta | bcc | $\langle 11\bar{1}\rangle\{101\}$, $\langle 11\bar{1}\rangle\{112\}$, $\langle 11\bar{1}\rangle\{123\}$ | 50 |

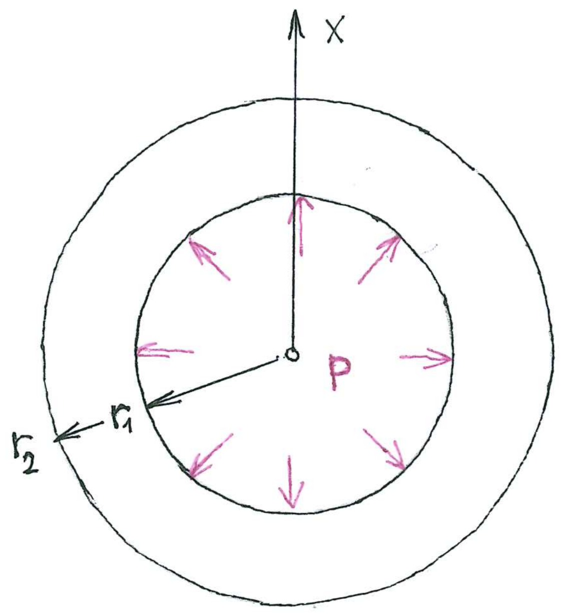


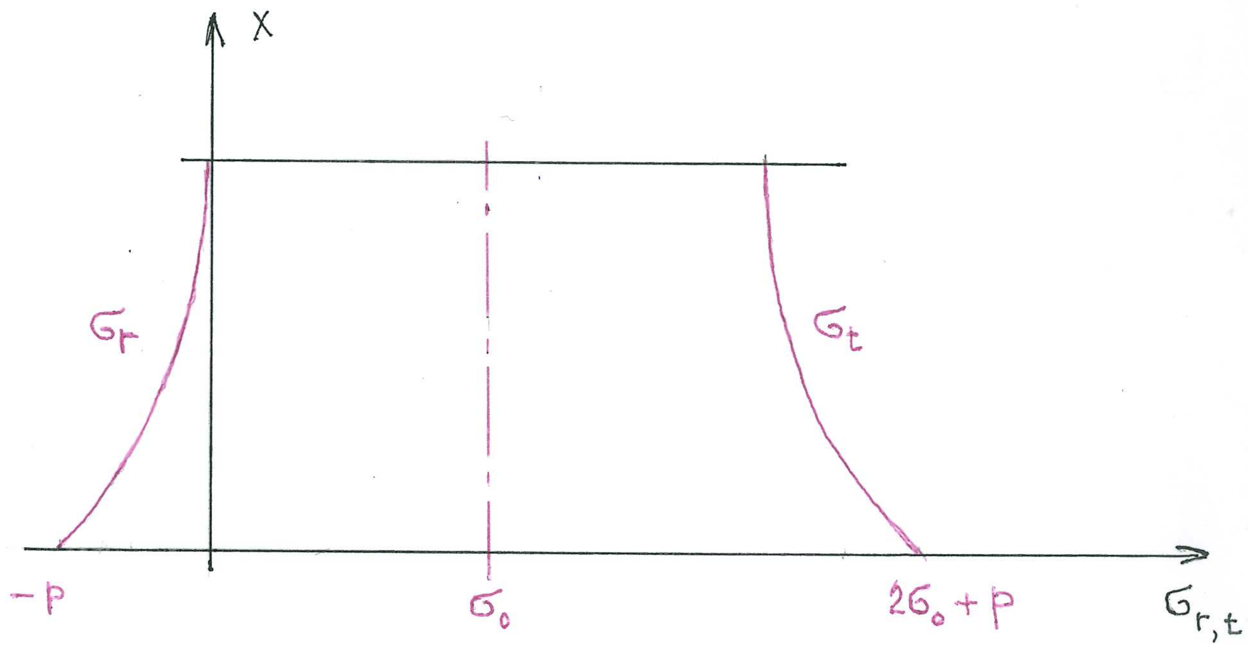




$$\boxed{\tau = \sigma \cos\phi \cos\lambda = \sigma(\mathbf{t}\cdot\mathbf{n})(\mathbf{t}\cdot\mathbf{s})}$$

Dunne, F., Petrinic, N. *Introduction to Computational Plasticity.* Oxford

# Thick wall tube - PMD Example Manual



# Thick wall tube (1/3)

Plastic state

$$\left.\begin{array}{lrcl} \text{equilibrium equation:} & \dfrac{\mathrm{d}\sigma_r}{\mathrm{d}x} + \dfrac{\sigma_r - \sigma_t}{x} & = & 0 \\ \text{yield condition:} & \tau_e = \sigma_t - \sigma_r & = & \sigma_Y \end{array}\right\} \Rightarrow \quad \frac{\mathrm{d}\sigma_r}{\mathrm{d}x} = \frac{\sigma_Y}{x}$$

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Hence

$$\sigma_r(x) = \sigma_Y \ln x - \sigma_Y \ln r_2 = \sigma_Y \ln \frac{x}{r_2}$$

# Thick wall tube (2/3)

Stress field at collapse

$$
\begin{aligned}
\sigma_r(x) &= \sigma_Y \ln \frac{x}{r_2} \\
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Critical pressure

$$\boxed{p = \sigma_Y \ln \frac{r_2}{r_1}}$$

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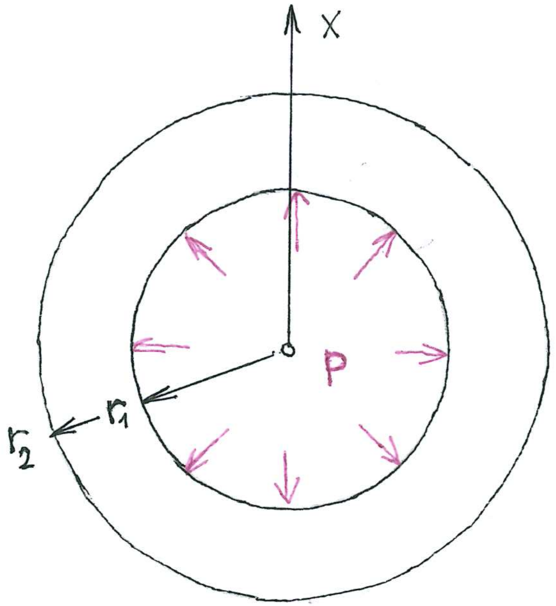


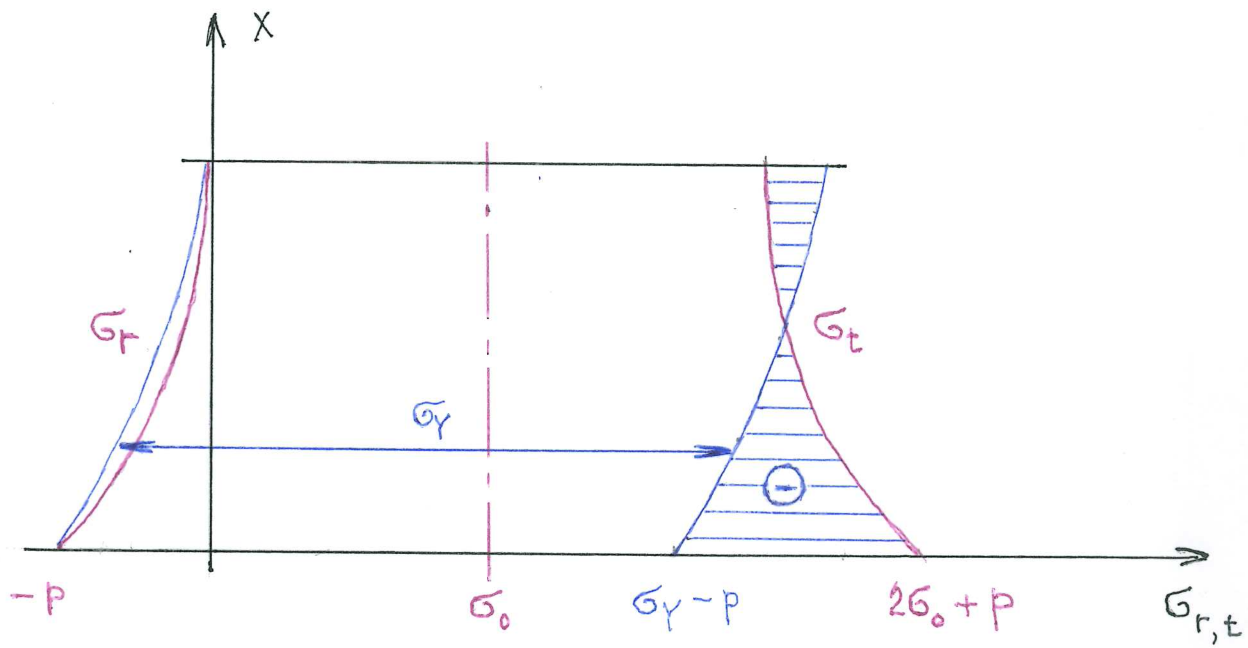


$$\sigma_r^{res}(r_1) = -p - (-p) = 0 \quad \Rightarrow \quad \tau_e^{res} = \sigma_t^{res} - \sigma_r^{res} = \sigma_t^{res} \quad (\text{at } r_1)$$

$$\sigma_t^{res}(r_1) = \sigma_Y - p - (2\sigma_0 + p) = \sigma_Y - 2(\sigma_0 + p)$$

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Shakedown condition

$$\boxed{\kappa = \frac{2r_2^2}{r_2^2 - r_1^2} \ln \frac{r_2}{r_1} \leq 2} \quad \Rightarrow \quad \frac{r_2}{r_1} < 2.2$$

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Applications: autofrettage, shot peening, laser shock peening.