Institute of Thermomechanics ASCR

YIELDING FUNCTIONS I

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Contents

Review

Yield function

$$\boxed{F(\sigma_{ij}) = f(\sigma_{ij}) - \sigma_Y \leq 0}$$

Noteworthy properties

- Convexity
- Hydrostatic sensitivity
- Tension versus compression

Deviatoric stress

Mean stress

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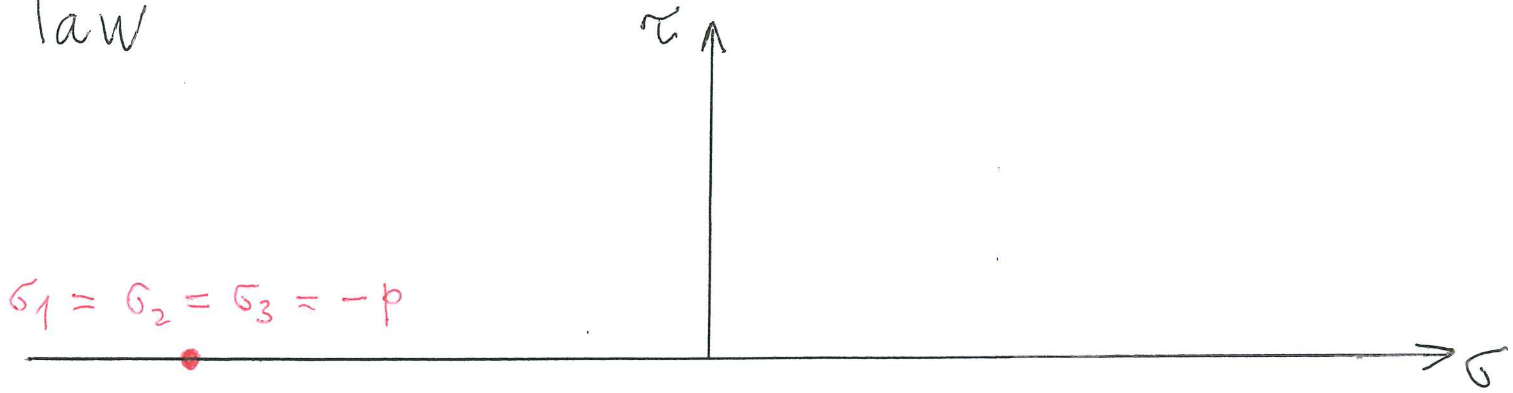
Deviatoric split

$$\boxed{F(\sigma_{ij}) = F(S_{ij}, \sigma_m)}$$

Pascal's law



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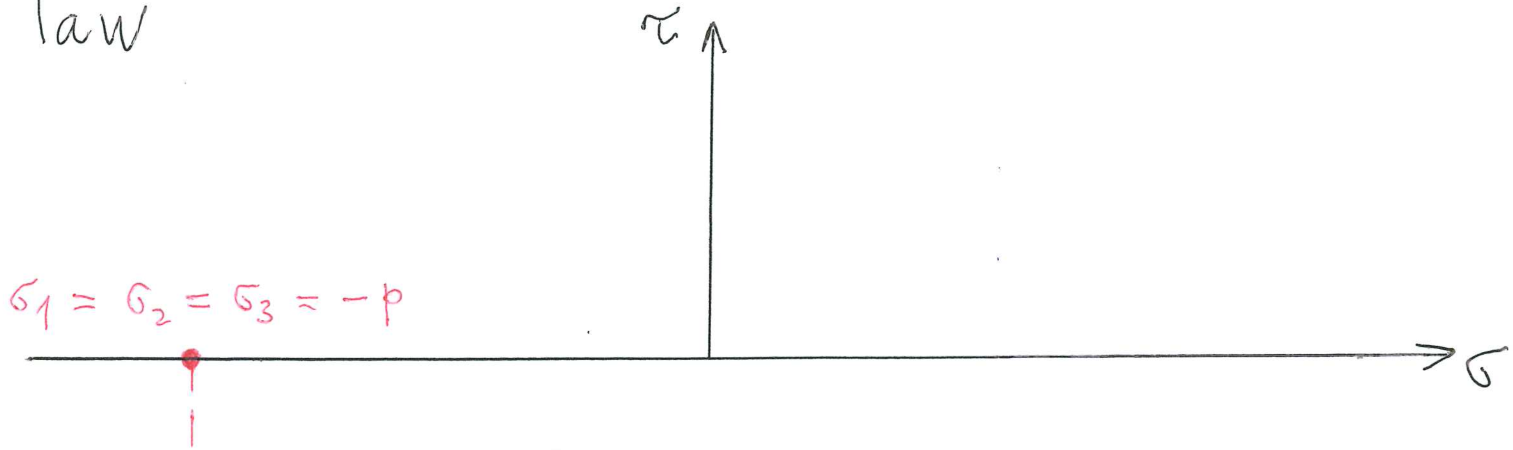
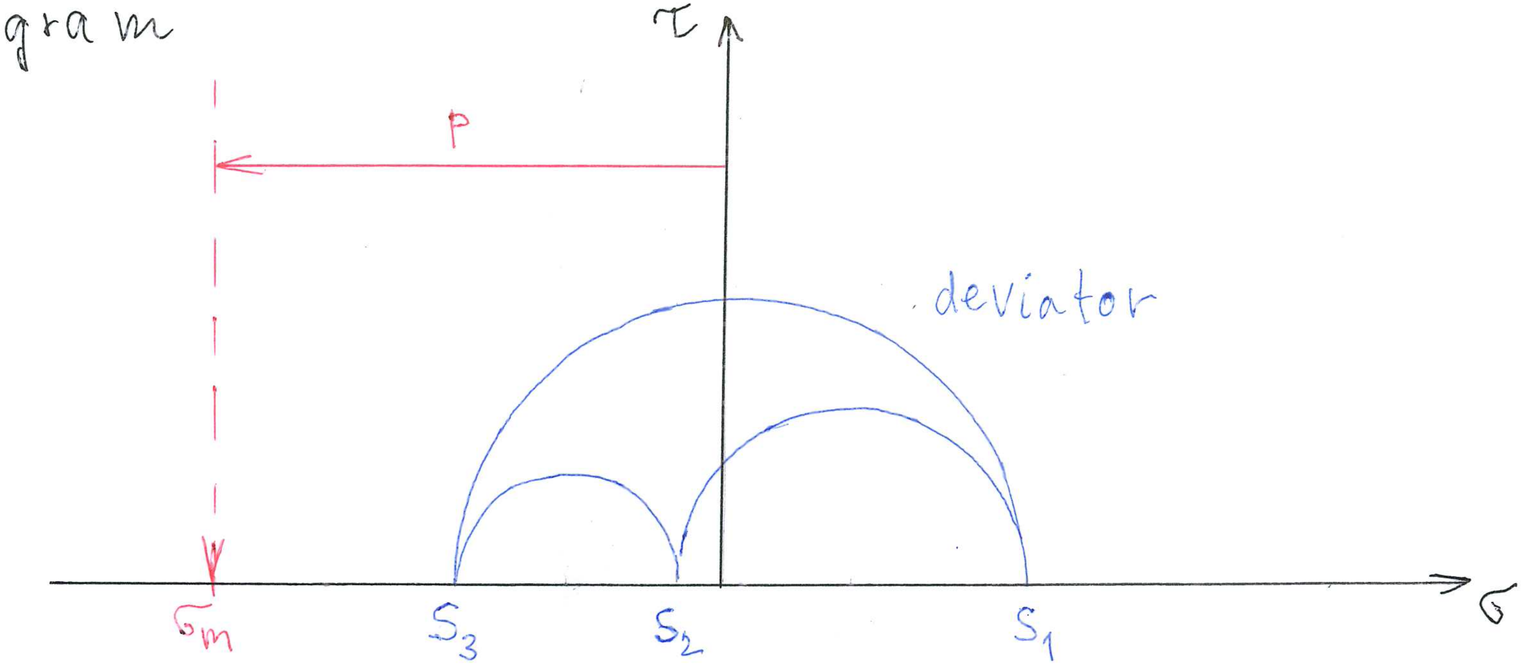
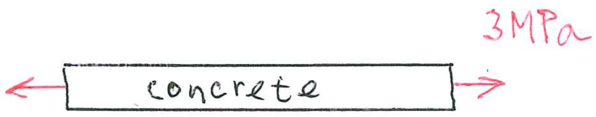
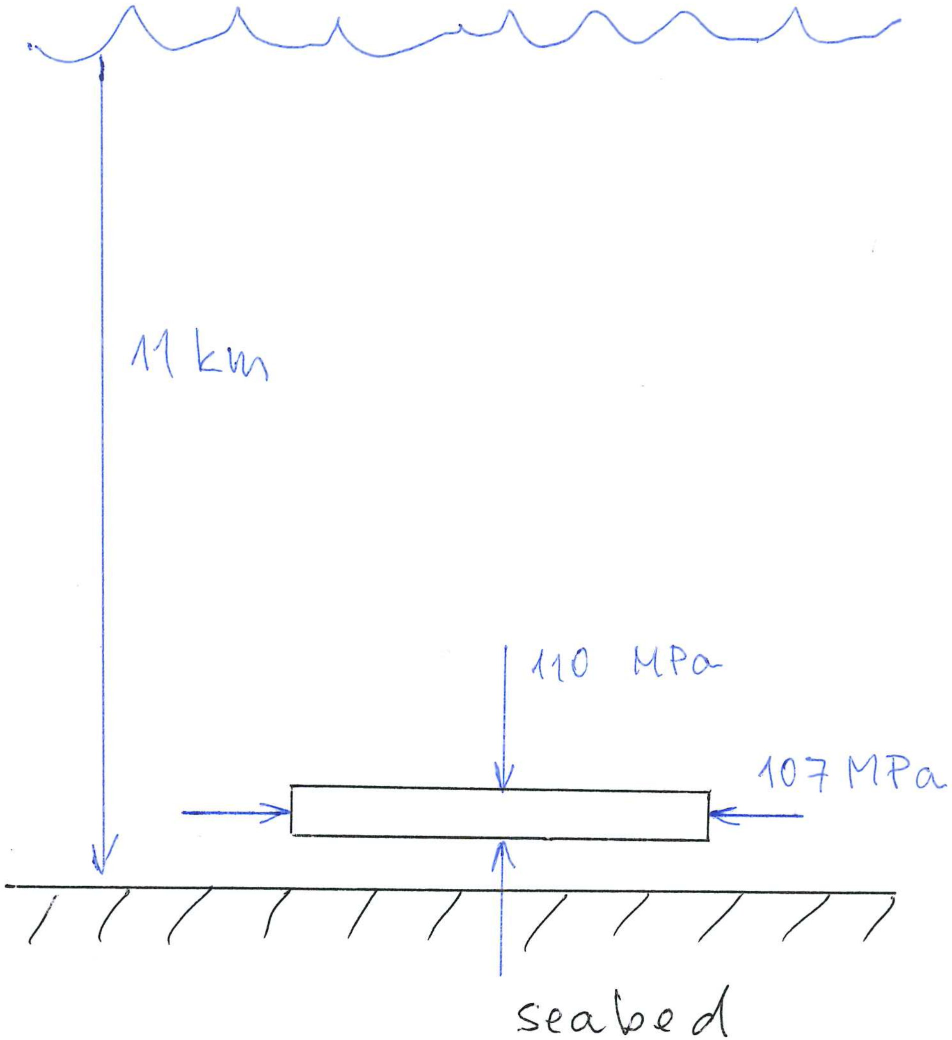


Mohr's diagram



Thought experiment





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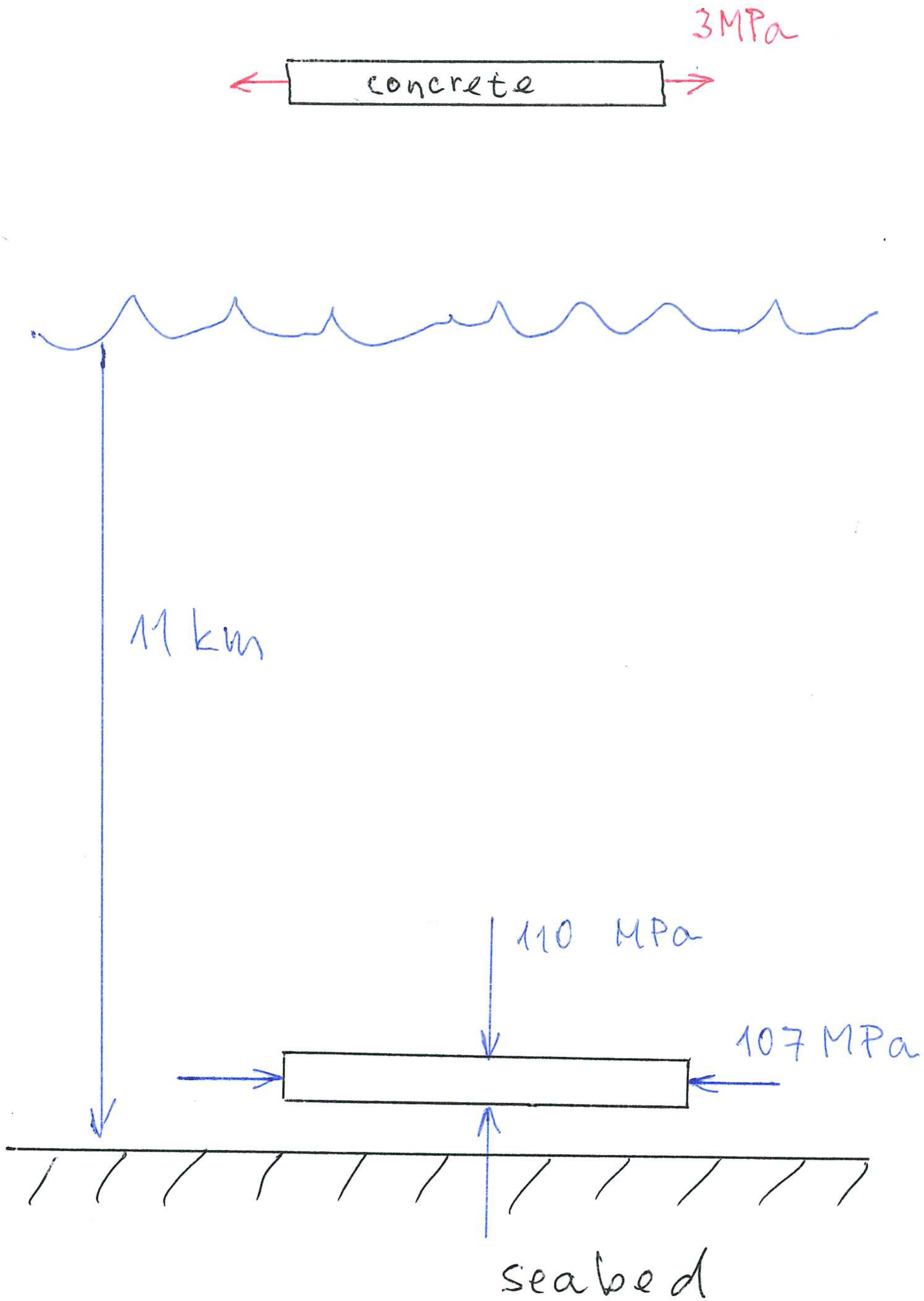

$$\underline{\sigma} = \begin{bmatrix} -107 & 0 & 0 \\ 0 & -110 & 0 \\ 0 & 0 & -110 \end{bmatrix}$$

$$\sigma_m = -109$$

$$\underline{s} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$s_m = 0 \quad \text{(checks)}$$

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Example: Tresca's effective stress

$$\tau_e = \sigma_1 - \sigma_3 = S_1 + \sigma_m - (S_3 + \sigma_m) = S_1 - S_3$$

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- Slip stress may or might not be influenced by the normal force (not shown in figure).

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5. Validation (next lecture)

Hydrostatic modification

Including the mean stress

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Note: The yield stress σ_Y is still a material constant (not variable).

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Resolved shear stress: $\tau_S = \dfrac{A_0\sigma}{A}\sin\phi$

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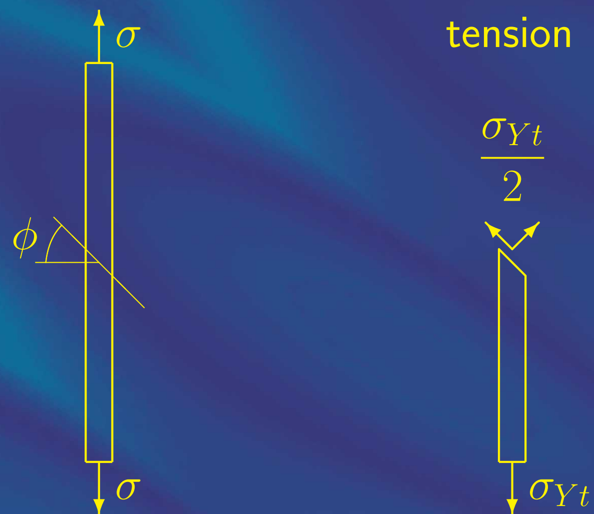

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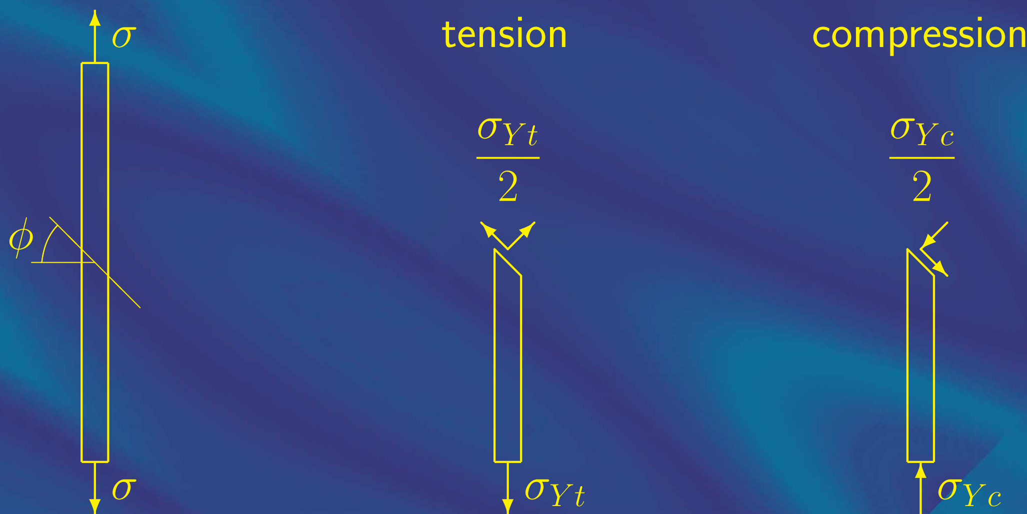

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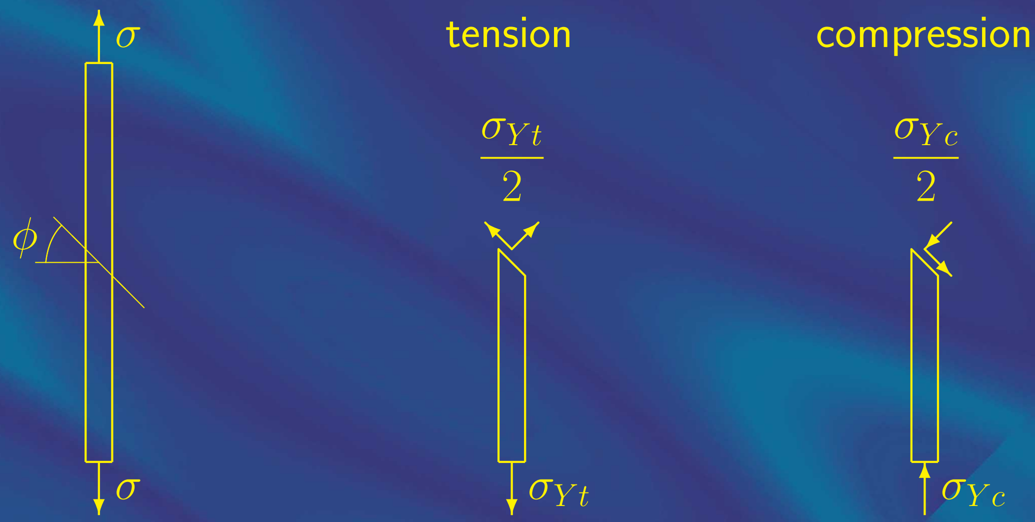

Conclusion: If normal stress plays no role, then $\sigma_{Yc} = \sigma_{Yt}$.

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Conclusion: The presence of normal stress decreases τ_{SY} for von Mises' condition.