ENGINEERING PLASTICITY II

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Contents

Review (1/2)

Limit loads

$$F_e \text{ (elastic limit)}, \quad F_p \text{ (plastic collapse)}, \quad \kappa = \frac{F_p}{F_e} \simeq 1.5$$

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Residual stress

$$\sigma^{\text{res}} = f(F) + g(-F)$$

f = elastic-plastic operator
g = elastic operator (assuming all elastic unloading)

$$-\sigma_{Yc} \leq \sigma^{\text{res}} \leq \sigma_{Yt} \qquad \text{(elastic range check)}$$

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Remark: $\mathcal{L}(F)$ may be regarded as fictitious stress as if no yielding had occurred.

Review (2/2)

For metals $\sigma_{Yc} = \sigma_{Yt} = \sigma_Y$. Then

$$\sigma^{\mathrm{res}}_{\mathrm{max}} = f(F_p) - \mathcal{L}(F_p) = f(F_p) - \kappa\mathcal{L}(F_e)$$

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Shakedown condition

$$\boxed{|\sigma^{\text{res}}_{\text{max}}| = \sigma_Y|1-\kappa| \le \sigma_Y \quad \Rightarrow \quad \kappa \le 2}$$

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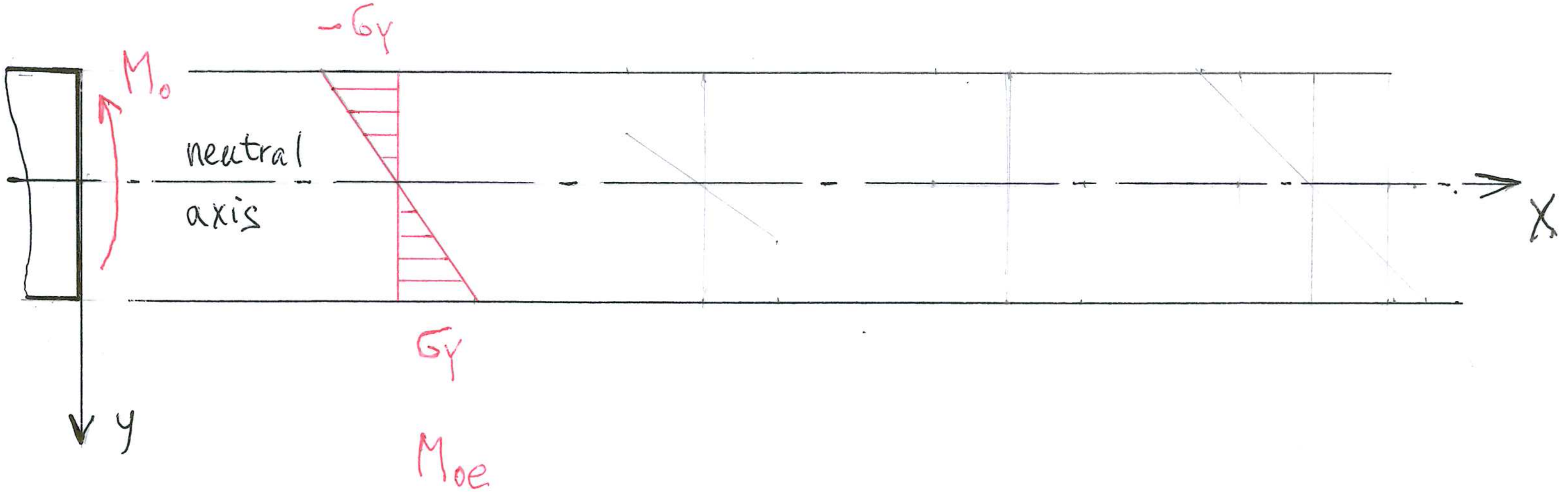
Static load assessment: $\sigma_D = \dfrac{\sigma_Y}{k'}, \quad k' = \dfrac{k}{\kappa} \simeq 1$

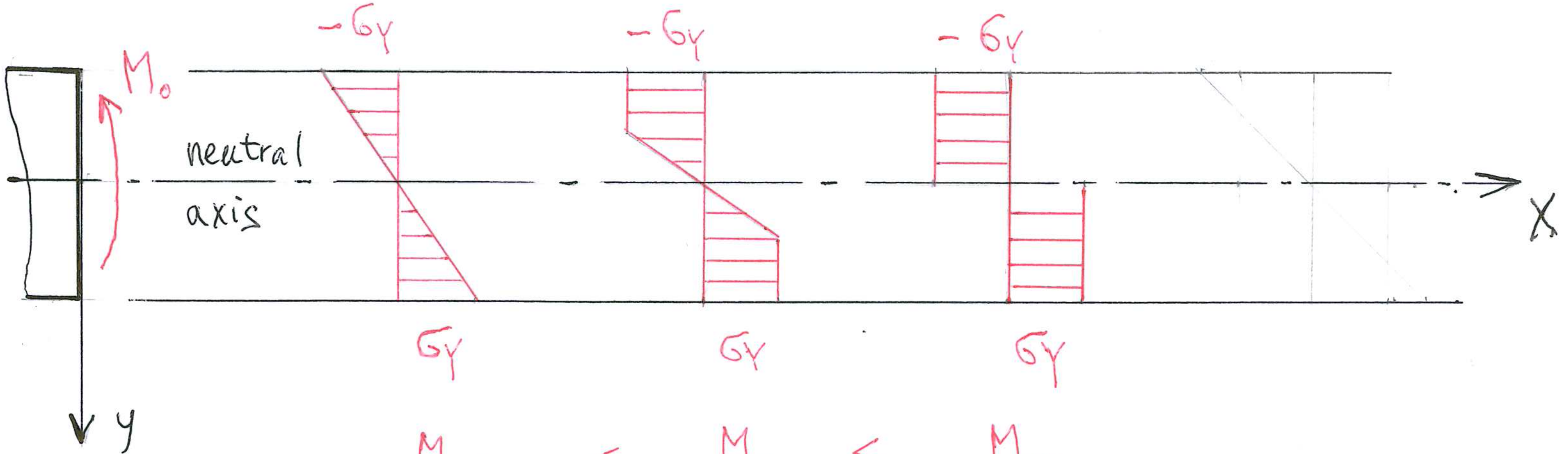
Life time prediction

- $\kappa \leq 2$ high cycle fatigue
- $\kappa > 2$ low cycle fatigue

Bending

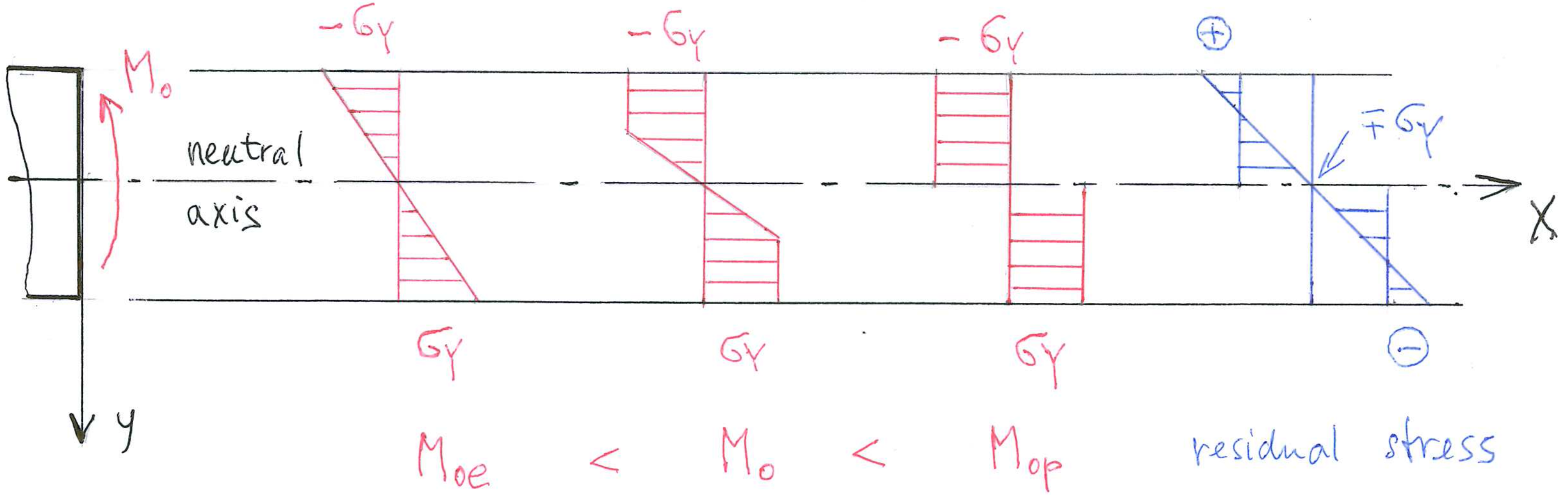


Bending



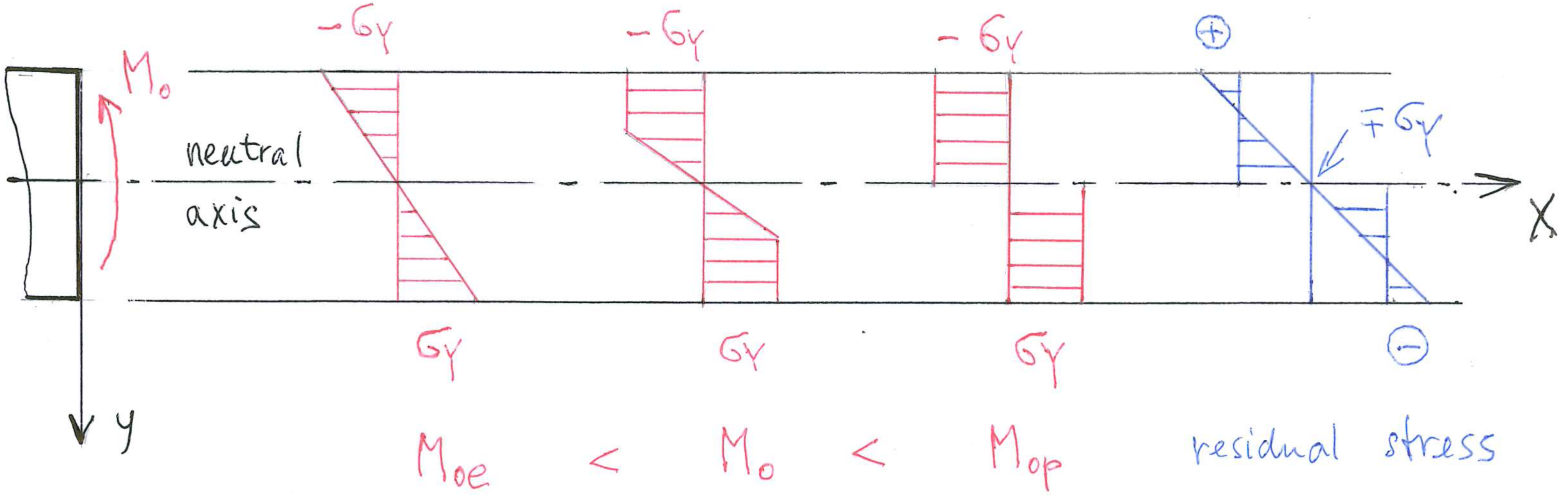
$$M_{oe} < M_o < M_{op}$$

Bending



M_o
neutral
axis
$-\sigma_Y$
$-\sigma_Y$
$-\sigma_Y$
$\oplus$
$\mp\sigma_Y$
X
σ_Y
σ_Y
σ_Y
$\ominus$
y
$M_{oe} < M_o < M_{op}$
residual stress

Bending



$$M_{op} = 2b\int_0^{h/2} y\,\sigma_Y\,dy = b\,\frac{h^2}{4}\,\sigma_Y \quad \Rightarrow \quad \boxed{W_{op} = \frac{bh^2}{4}}$$

Bending

Shape factor

$$\kappa_o = \frac{M_{op}}{M_{oe}} = \frac{W_{op}\sigma_Y}{W_o\sigma_Y} = \frac{W_{op}}{W_o} = \frac{6}{4} = 1.5$$

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Beam's bottom fibre

$$\sigma^{\text{res}}\left(\frac{h}{2}\right) = \sigma_Y - \frac{M_{op}}{W_o} = \sigma_Y - \frac{W_{op}\sigma_Y}{W_o} = \sigma_Y(1-\kappa_o)$$

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Quizz

- $\kappa_o = ?$ for I-profile
- $\kappa_o = ?$ for $\bigcirc$

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- $\kappa_o = ?$ for $\bigcirc$

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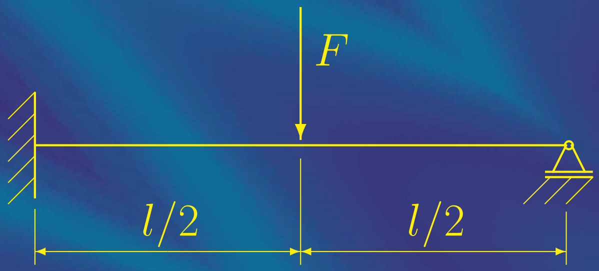
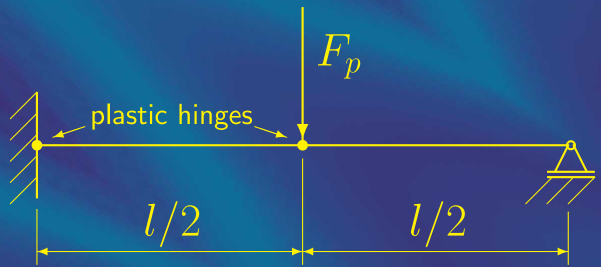
Quizz

- $\kappa_o = 1.15$ for I-profile
- $\kappa_o = 1.7$ for $\bigcirc$

Principle of virtual work



Principle of virtual work



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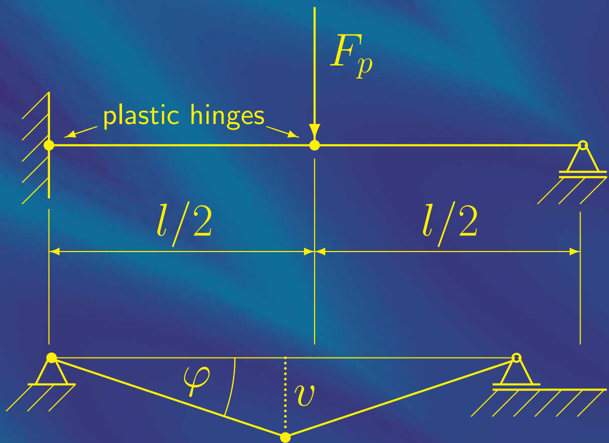

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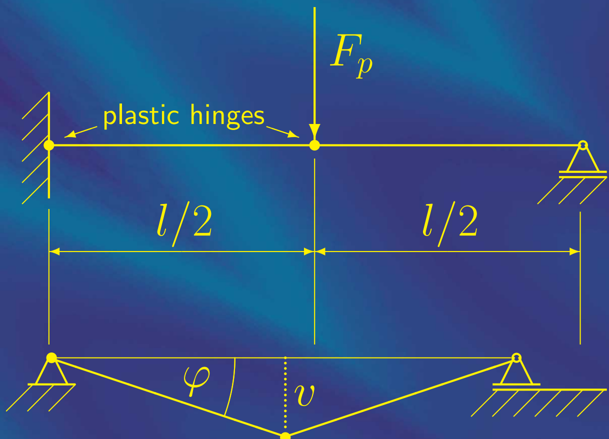
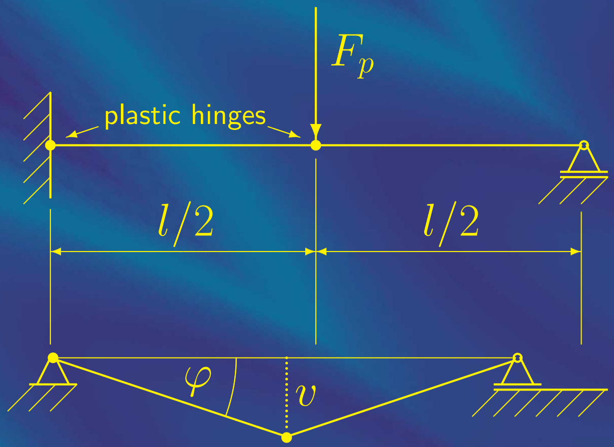


Collapse mode

$$\varphi \simeq \sin\varphi = \frac{v}{l/2} = \frac{2v}{l}$$

Principle of virtual work



Dissipation

$$\mathcal{D} = M_{op}\varphi + 2M_{op}\varphi = 3M_{op}\varphi$$

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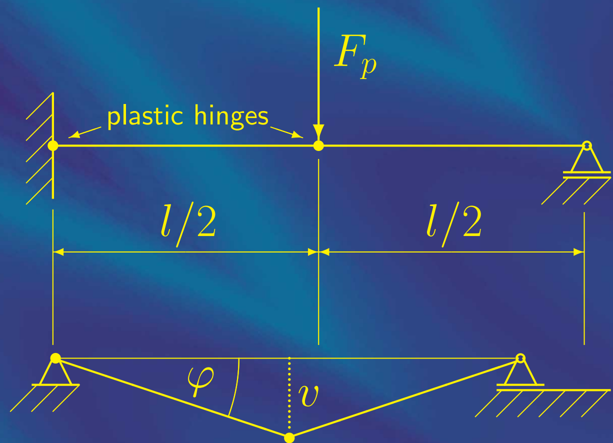

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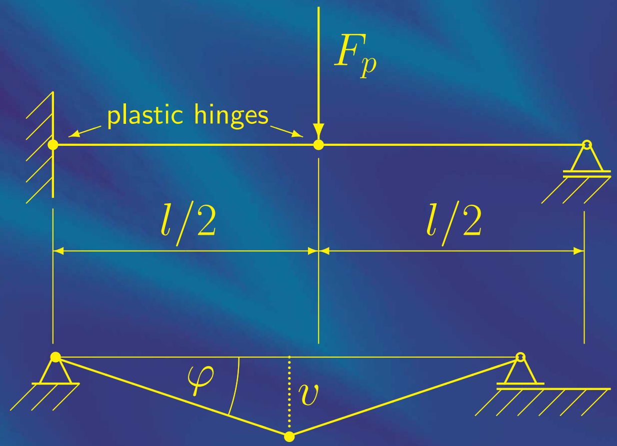

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comparing . . .

$$\boxed{F_p = \frac{6M_{op}}{l}}$$

Exercise: Testing various boundary conditions is quite easy for the PVW method.

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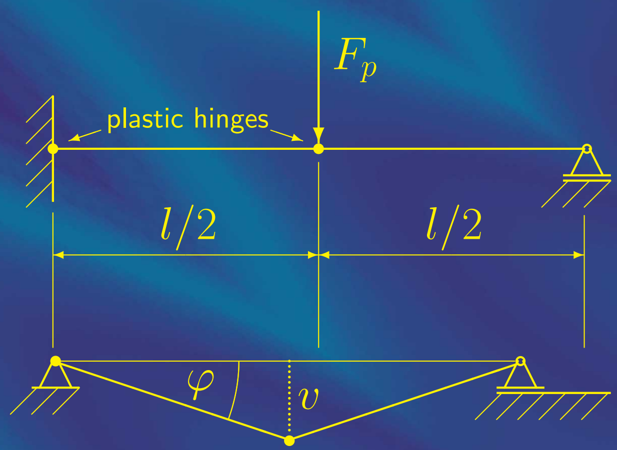

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Remark: Equilibrium equation not needed!

Shakedown assessment

Elastic solution: $M_o^{\mathrm{max}} = 3Fl/16$

$$\boxed{F_e = \frac{16 M_{oe}}{3l}}$$

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Elastic-plastic load factor

$$\kappa = \frac{F_p}{F_e} = \frac{6M_{op}}{16M_{oe}/3} = \frac{9}{8}\kappa_o$$

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Circular cross-section: $\kappa_o = 1.7 \quad \Rightarrow \quad \kappa = 1.9 < 2 \quad \Rightarrow$ shakedown will occur

$$\boxed{F_{\text{shakedown}} = F_p}$$

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Remark: In general $F_{\text{shakedown}} \leq F_p$. Shakedown factor $= F_{\text{shakedown}}/F_{\text{allowed}}$

Terminology

Load factor

$$k = \frac{F_p}{F_{\text{allowed}}}$$

Shakedown factor

$$k_s = \frac{F_{\text{shakedown}}}{F_{\text{allowed}}}$$

Elastic-plastic load (shape) factor

$$\kappa = \frac{F_p}{F_e}$$

Factor of safety

$$k' = \frac{k}{\kappa} = \frac{\sigma_Y}{\sigma_D}$$

Torsion

Stress function

$$\exists\, \phi(y,z) : \begin{cases} \tau_y &= -\dfrac{\partial \phi}{\partial y} \\[2ex] \tau_z &= \dfrac{\partial \phi}{\partial z} \end{cases}$$

Equilibrium eqn. automatically satisfied.

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$$\boxed{M_k = 2 \int_A \phi \, \mathrm{d}S}$$

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Exercise: Verify the general solution for a special case of circular cross-section.

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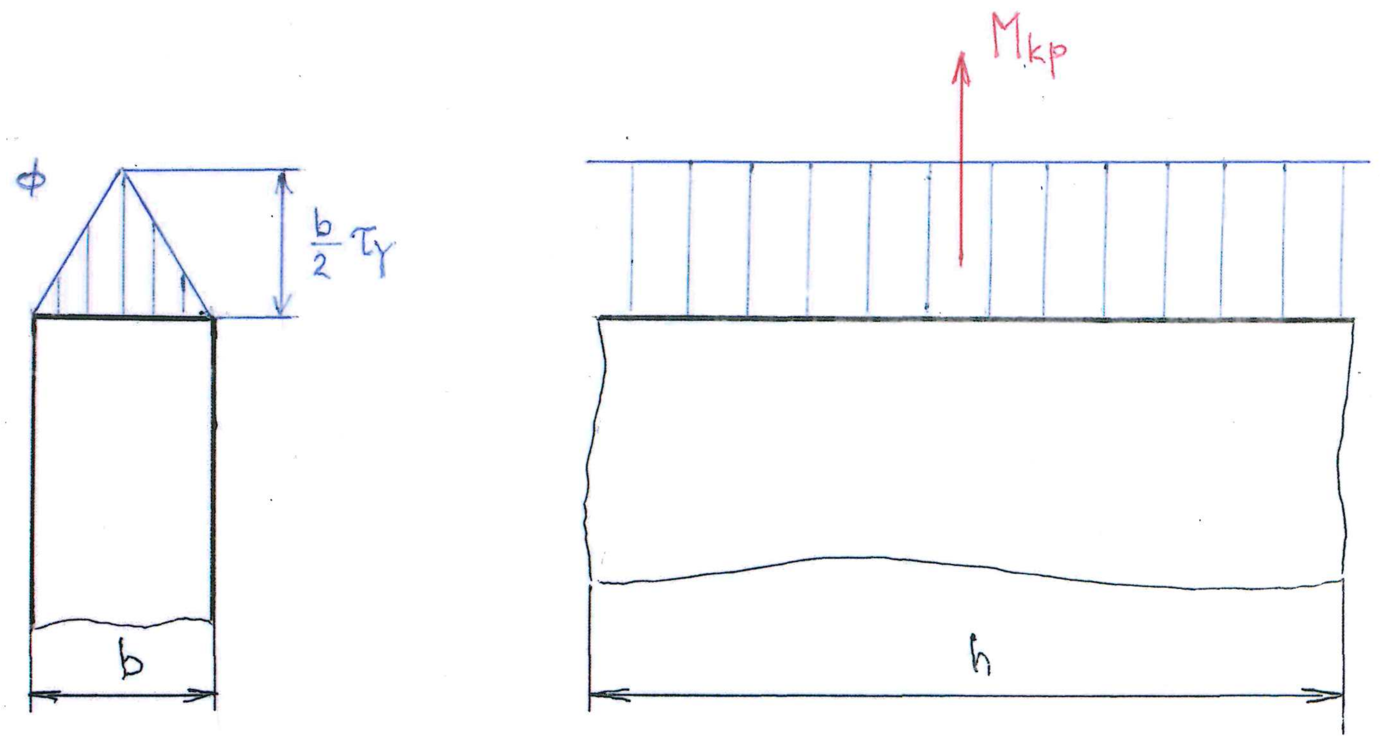

$$M_{kp} = 2\,\frac{1}{2}\,b\left(\frac{b}{2}\tau_Y\right)\cdot h = \frac{b^2 h}{2}\tau_Y \quad \Rightarrow \quad \boxed{W_{kp} = \frac{b^2 h}{2}}$$

$$\varkappa_k = \frac{M_{kp}}{M_{ke}} = \frac{W_{kp}}{W_k} = \frac{3}{2} = 1.5$$