HARDENING

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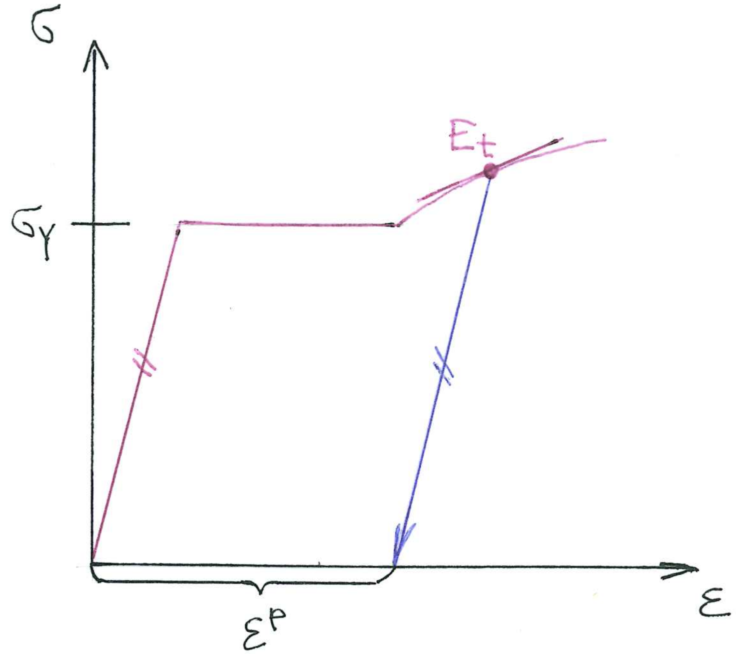
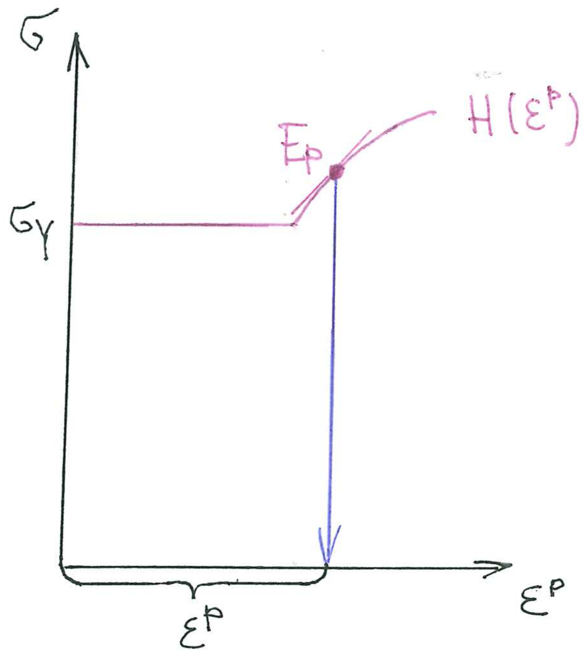
uniaxial hardening





Plastic modulus

Hardening function

$$H(\epsilon^p) \geq \sigma_Y \ , \quad H(0) = \sigma_Y \ , \quad E_p = H'(\epsilon^p)$$

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It holds that $E_t < E$ and $E_p \to \infty$ as $E_t \to E$, which causes some ill-conditioning for smooth σ-ϵ curves. Progressive hardening $E_t > E$ is prohibited. Perfect plasticity is recovered for $E_t = 0$.

Equivalent plastic strain

Effective strain rate

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$$\epsilon_p = \int_0^t \dot{\epsilon}_p \, \mathrm{d}\tau \qquad \text{(history dependent)}$$

Remark: The latter quantity is also known as the cumulative plastic strain.

Isotropic hardening (1/2)

Hill (1950)

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Substitution

$$\dot{\sigma}_{ij} = C_{ijkl}(\dot{\epsilon}_{kl} - \lambda R_{kl}) \quad \text{and} \quad \dot{\epsilon}_p = \lambda \sqrt{\tfrac{2}{3} R_{mn} R_{mn}}$$

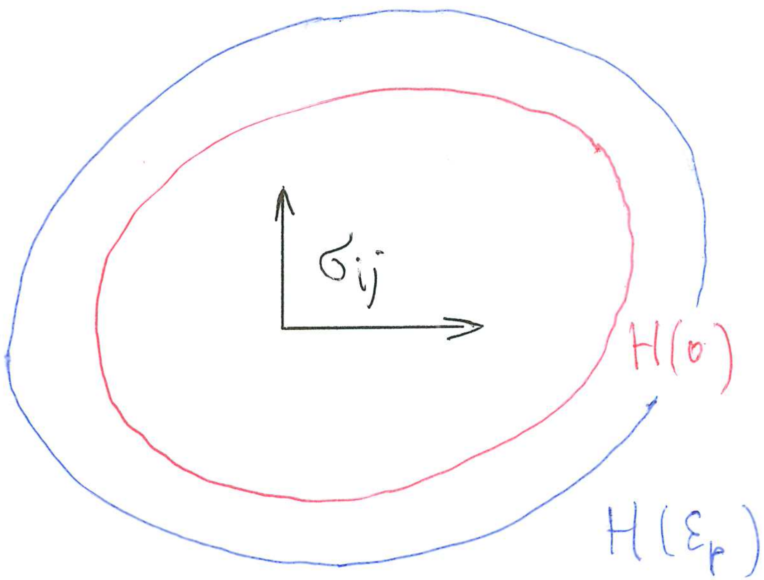
Isotropic hardening (2/2)

Plastic multiplier

$$\lambda = \frac{\dfrac{\partial F}{\partial \sigma_{ij}} C_{ijkl} \dot{\epsilon}_{kl}}{E_p \sqrt{\frac{2}{3} R_{mn} R_{mn}} + \dfrac{\partial F}{\partial \sigma_{pq}} C_{pqrs} R_{rs}}$$

Perfect plasticity is recovered, setting $E_p = 0$.

<u>Isotropic hardening</u>



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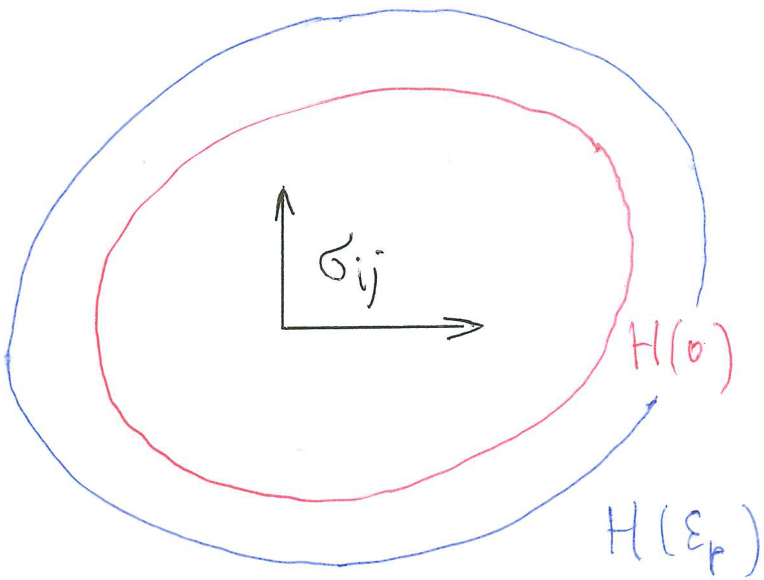
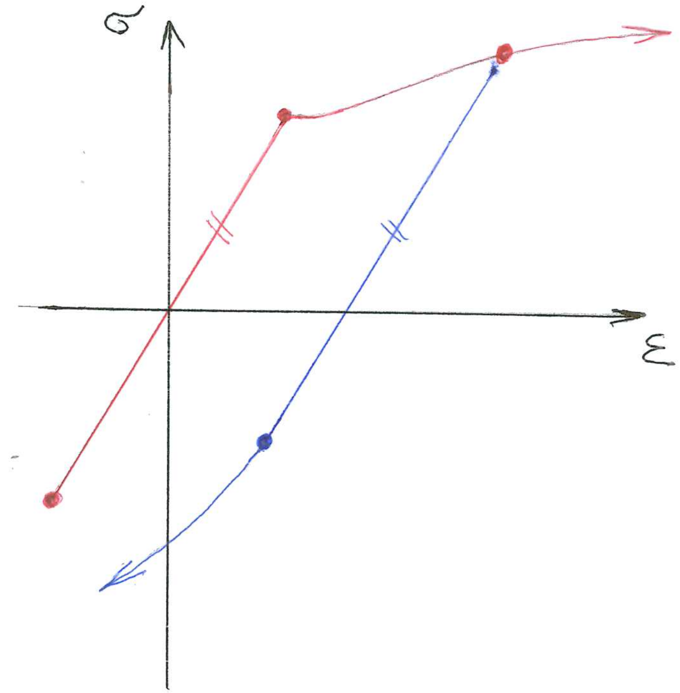
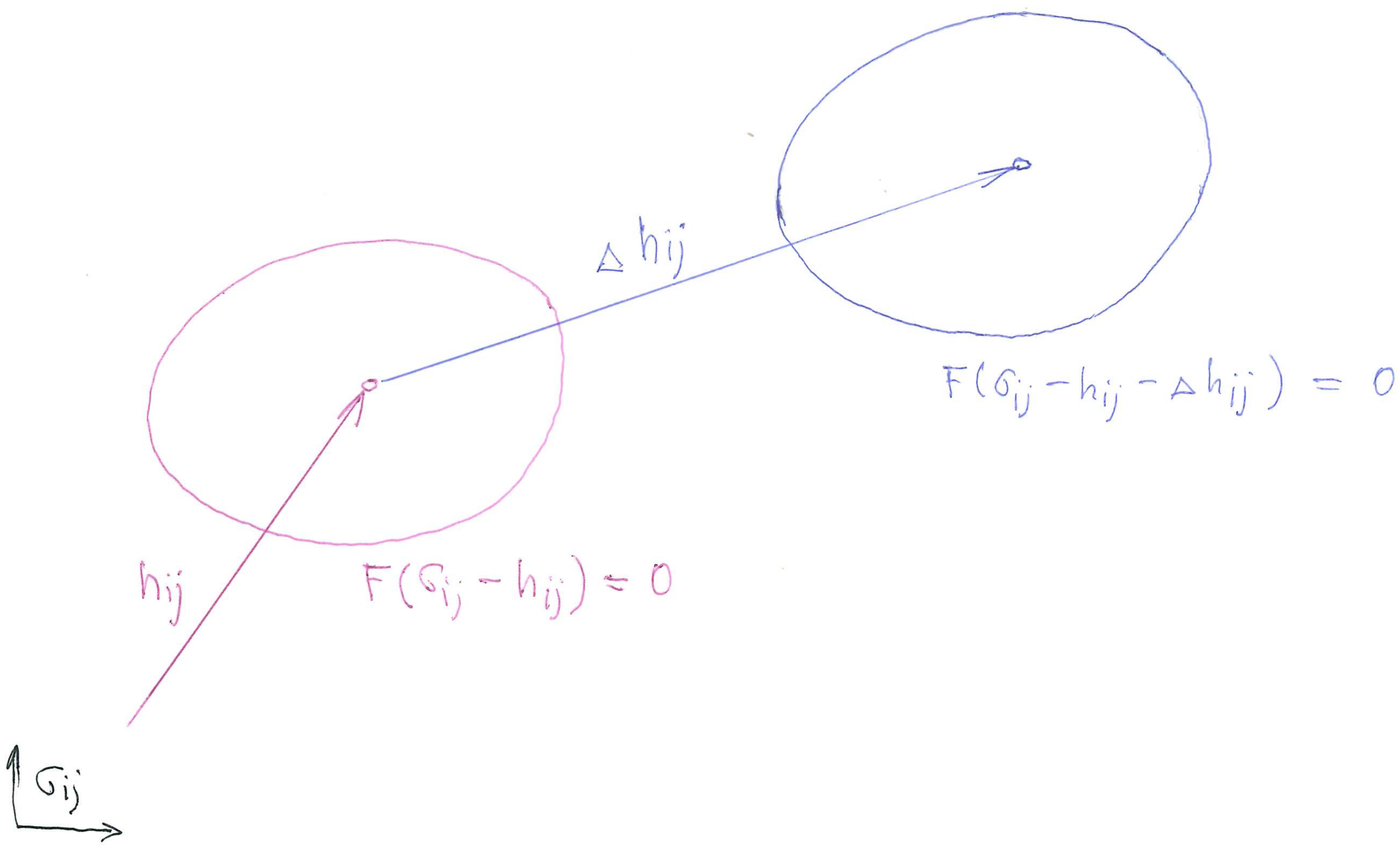


Bauschinger effect (1890)



Elastic range approximately preserved.

Kinematic hardening



Kinematic hardening (1/3)

Prager (1955)

$$\boxed{f(\sigma_{ij} - h_{ij}) \leq \sigma_Y}$$

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Constitutive assumption

$$\boxed{h_{ij} = c\epsilon^p_{ij}\,, \qquad c = \text{const.}}$$

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Plastic multiplier

$$\lambda = \frac{\dfrac{\partial F}{\partial \sigma_{ij}} C_{ijkl} \dot{\epsilon}_{kl}}{c \dfrac{\partial F}{\partial \sigma_{mn}} R_{mn} + \dfrac{\partial F}{\partial \sigma_{pq}} C_{pqrs} R_{rs}}$$

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It must hold that $\lambda_{\text{kin}} = \lambda_{\text{iso}}$, therefore

$$c \frac{\partial F}{\partial \sigma_{mn}} R_{mn} = E_p \sqrt{\tfrac{2}{3} R_{mn} R_{mn}}$$

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Associated flow rule

$$c \sqrt{\frac{\partial F}{\partial \sigma_{mn}} \frac{\partial F}{\partial \sigma_{mn}}} = \sqrt{\frac{2}{3}} E_p$$

Kinematic hardening (3/3)

J_2-theory

$$\frac{\partial F}{\partial \sigma_{mn}}\frac{\partial F}{\partial \sigma_{mn}} = \frac{3}{2}\frac{S_{mn}}{\sigma_e}\frac{3}{2}\frac{S_{mn}}{\sigma_e} = \frac{3}{2}$$

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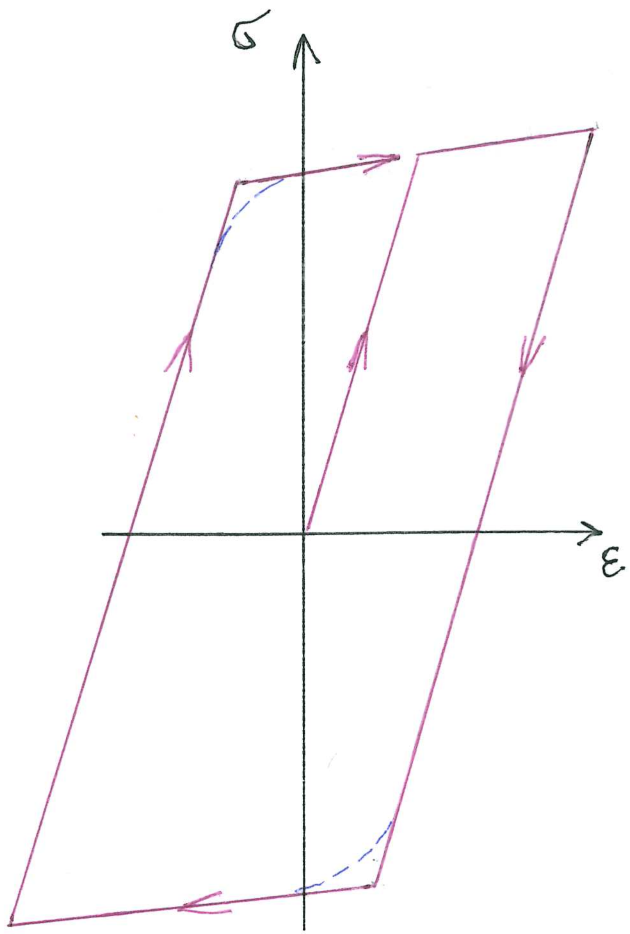
Important note: Since c is a constant, so is E_p, which implies only linear hardening is available for this particular model!

Closed kinematic cycle



(hysteretic loop)

Isotropic

+ arbitrary $H(\varepsilon_p)$

− unloading

Kinematic

+ cyclic plasticity

− bilinear diagram