Institute of Thermomechanics ASCR

LIMIT LOAD ANALYSIS I

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Contents

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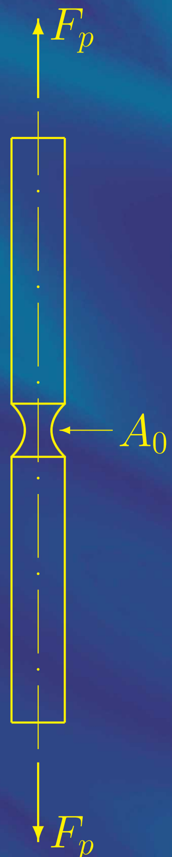
Motivation



Admissible stress method

$$F_p = A_0\,\sigma_Y$$

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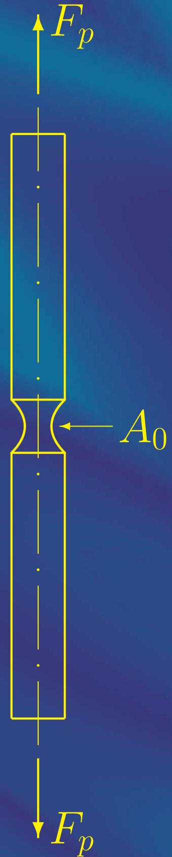

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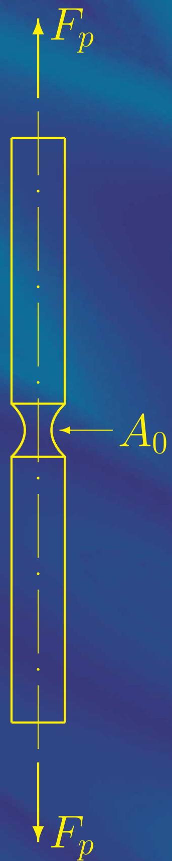

Admissible stress method

$$F_p \geq A_0\,\sigma_Y$$

Principle of virtual work

$$F_p \leq \text{PVW estimate}$$

Problem definition

Proportional loading

$$\mathbf{b} = \alpha \mathbf{b}^0 , \quad \mathbf{t} = \alpha \mathbf{t}^0$$

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Assumptions

- Perfect plasticity.
- Yield function is convex.
- Associated flow rule.

Weak formulation

SAS: Statically admissible set $\{\boldsymbol{\sigma}, \mathbf{b}, \mathbf{t}\}$ satisfies

$$\sigma_{ij,j} + b_i = 0 \text{ in } \Omega \qquad \text{and} \qquad \sigma_{ij} n_j = t_i \text{ on } \Gamma$$

We relax on $F(\sigma_{ij}) \leq 0$ at this point.

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Remark: The principle of virtual work can now be derived.

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But $C_{ijkl}\dot{\epsilon}^e_{kl}\dot{\epsilon}^e_{ij} \geq 0$, therefore $\dot{\epsilon}^e_{ij} \equiv 0 \quad \Rightarrow \quad \dot{\sigma}_{ij} \equiv 0$ in Ω.

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$$\int_\Omega \sigma^*_{ij}\dot{\epsilon}_{ij}\,\mathrm{d}V = \alpha^*_p \int_\Omega b^0_i \dot{u}_i\,\mathrm{d}V + \alpha^*_p \int_\Gamma t^0_i \dot{u}_i\,\mathrm{d}S$$

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Plastic zone

$$\Omega_p \equiv \{\forall \mathbf{x} \in \Omega \,|\, \dot{\boldsymbol{\epsilon}}^p(\mathbf{x}) \neq 0\}$$

Admissible stress method (3/3)

Thus

$$\frac{\alpha_p}{\alpha_p^*} = \frac{\displaystyle\int_{\Omega_p} \sigma_{ij}\dot{\epsilon}_{ij}^p \,\mathrm{d}V}{\displaystyle\int_{\Omega_p} \sigma_{ij}^*\dot{\epsilon}_{ij}^p \,\mathrm{d}V}$$

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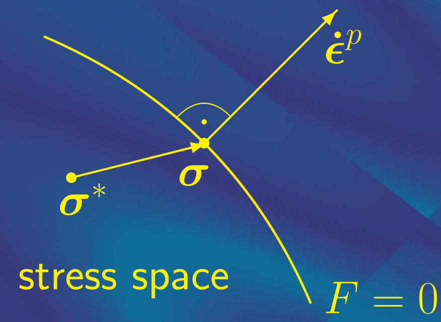

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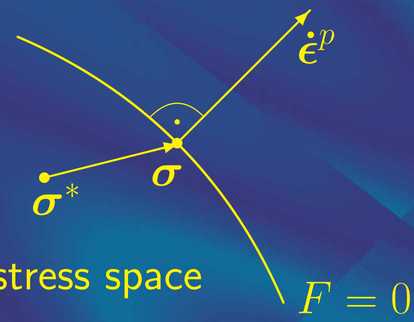

stress space

convexity condition:

$$(\sigma_{ij} - \sigma_{ij}^*)\dot{\epsilon}_{ij}^p \geq 0$$

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$$\mathcal{D} = \dot{W}^* = \alpha_p^* \int_\Omega b_i^0 \dot{u}_i^* \,\mathrm{d}V + \alpha_p^* \int_\Gamma t_i^0 \dot{u}_i^* \,\mathrm{d}S \quad \Rightarrow \quad \alpha_p^*$$

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where the integration domain could safely be reduced to Ω^*_p.

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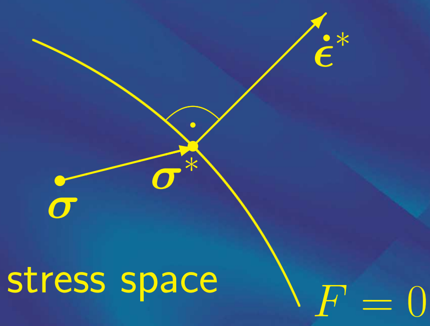

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