Institute of Thermomechanics ASCR

IMPLEMENTATION IN FEM

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- Weak formulation
- Solution algorithm
- Numerical example
- Locking
- Consistent operator

Weak formulation (1/2)

Principle of virtual work

$$\int_V \boldsymbol{\sigma}^T \delta \boldsymbol{\epsilon} \, \mathrm{d}V = \int_V \delta \boldsymbol{\epsilon}^T \boldsymbol{\sigma} \, \mathrm{d}V = \delta \mathbf{u}^T \mathbf{R}$$

Weak formulation (1/2)

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Strain-displacement matrix

$$\boldsymbol{\epsilon} = \mathbf{B}\mathbf{u} \quad \Rightarrow \quad \delta\boldsymbol{\epsilon} = \mathbf{B}\,\delta\mathbf{u}$$

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Nodal equilibrium

$$\boxed{\int_V \mathbf{B}^T\boldsymbol{\sigma}\,\mathrm{d}V = \mathbf{R}}$$

Weak formulation (2/2)

Elasticity

$$\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\epsilon} = \mathbf{C}\mathbf{B}\mathbf{u}$$

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...inserting into equilibrium equation

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$$\underbrace{\left(\int_V \mathbf{B}^T \mathbf{C}^{\mathsf{ep}} \mathbf{B} \, \mathrm{d}V \right)}_{\mathbf{K}_T} \dot{\mathbf{u}} = \dot{\mathbf{R}}$$

Remark : $\mathbf{K}_T$ symmetric only for the associated flow rule!

Solution algorithm

INPUT: $\mathbf{u}_0 =$ elastic solution

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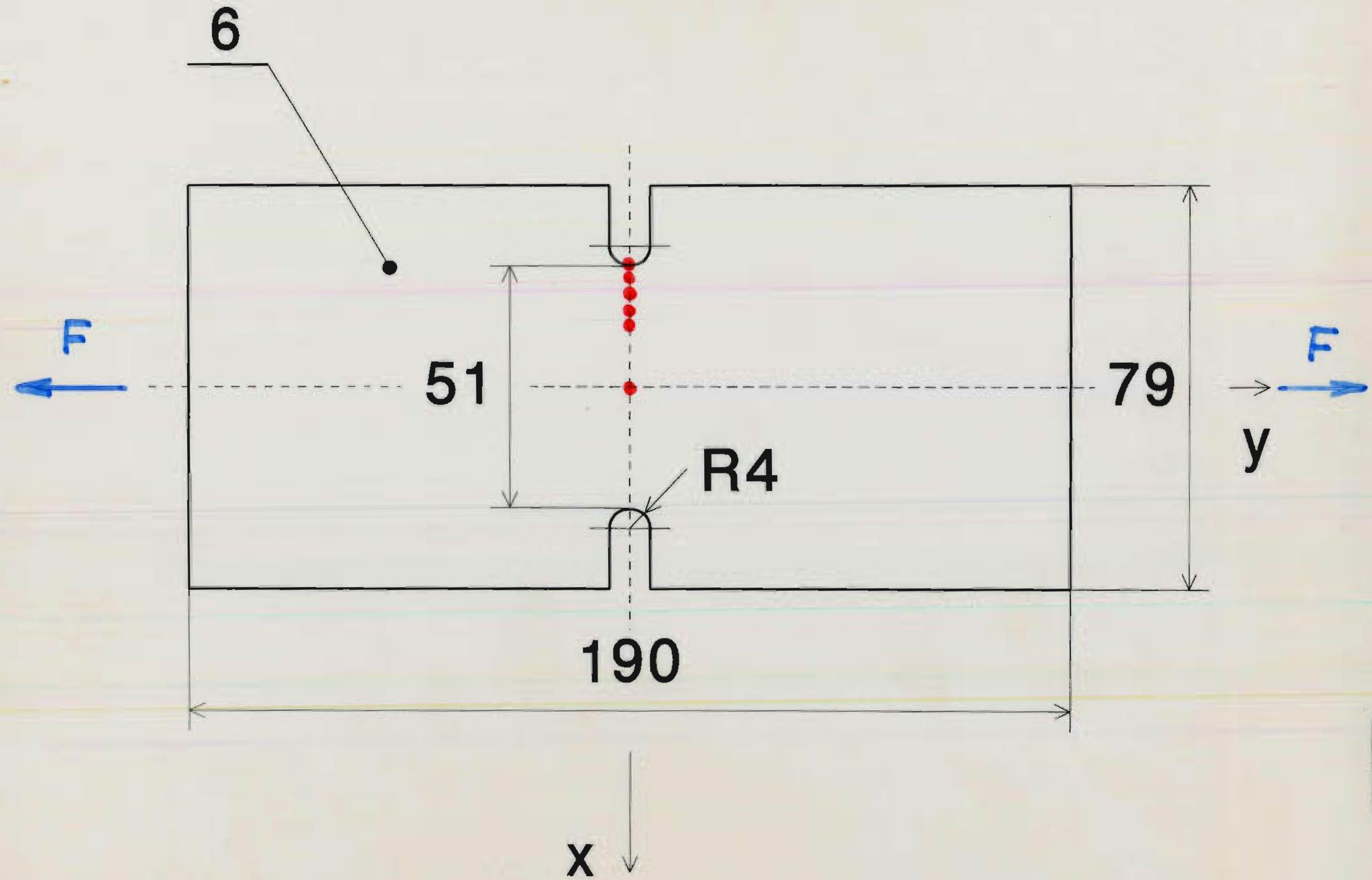
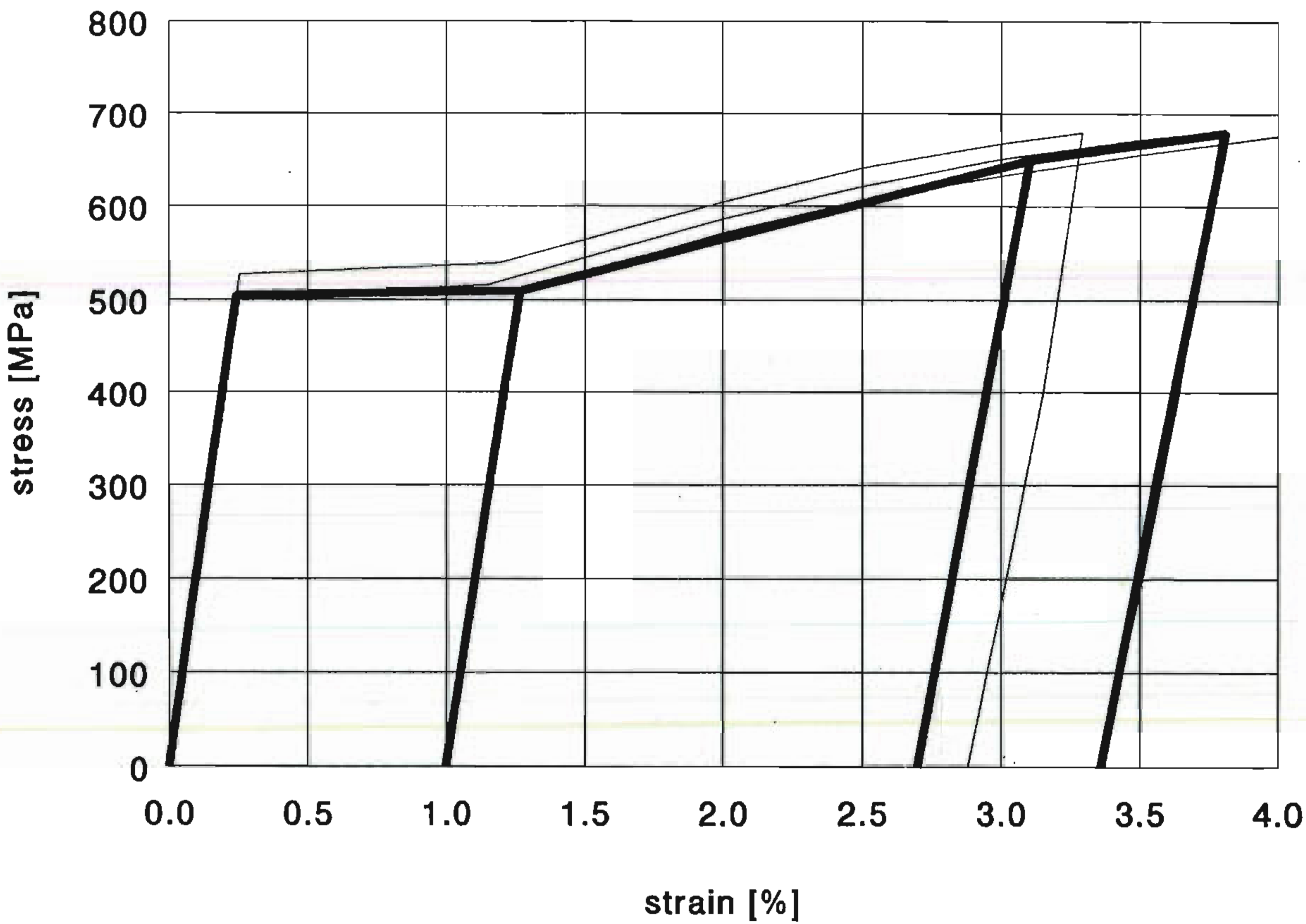
Remarks:

- Symmetry of $\mathbf{K}_T$ is required by the Newton-Raphson procedure.
- BFGS (Broyden-Fletcher-Goldfarb-Shanno) method is extremely effective.

Numerical example

Plešek, J.: Numerical analysis of a notched inelastic specimen and comparison with experimental results. Comp. Struct., **48**, No. 3, pp. 523–528, 1993.


6
F
51
79
F
y
R4
190
x



800
700
600
500
400
300
200
100
0
stress [MPa]
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
strain [%]

Notched specimen

Importance of stress concentration $\alpha = 3$ (same as for a hole) in engineering:

$$\sigma_{\text{max}}^{\text{res}} = \sigma_Y - \alpha\sigma_D^{\text{nom}} = \sigma_Y - \alpha\frac{\sigma_Y}{k} = \sigma_Y(1 - \frac{\alpha}{k})$$

since the stress state is nearly uniaxial at the root of the notch.

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$$\kappa = \frac{\alpha}{k} \simeq 2$$

narrowly meeting the shakedown condition.

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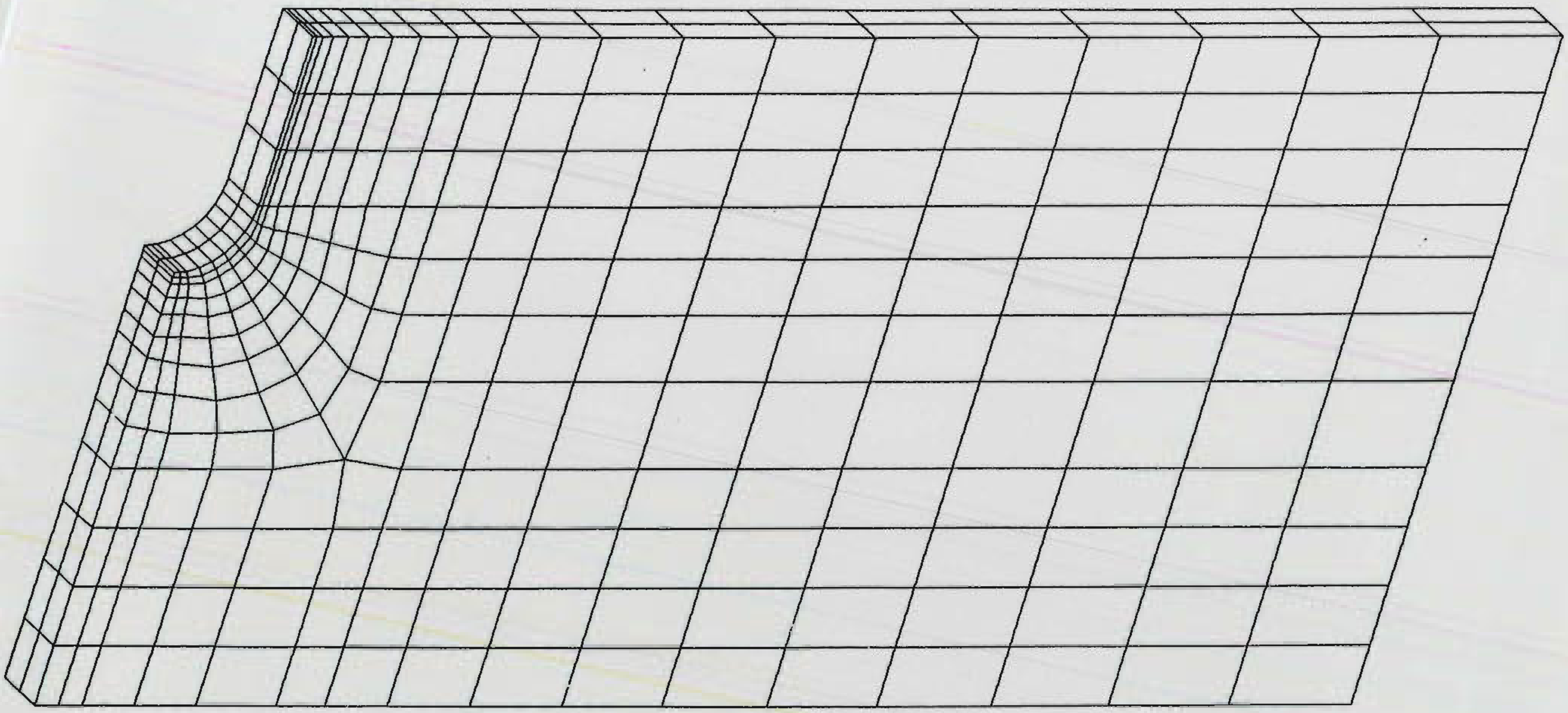
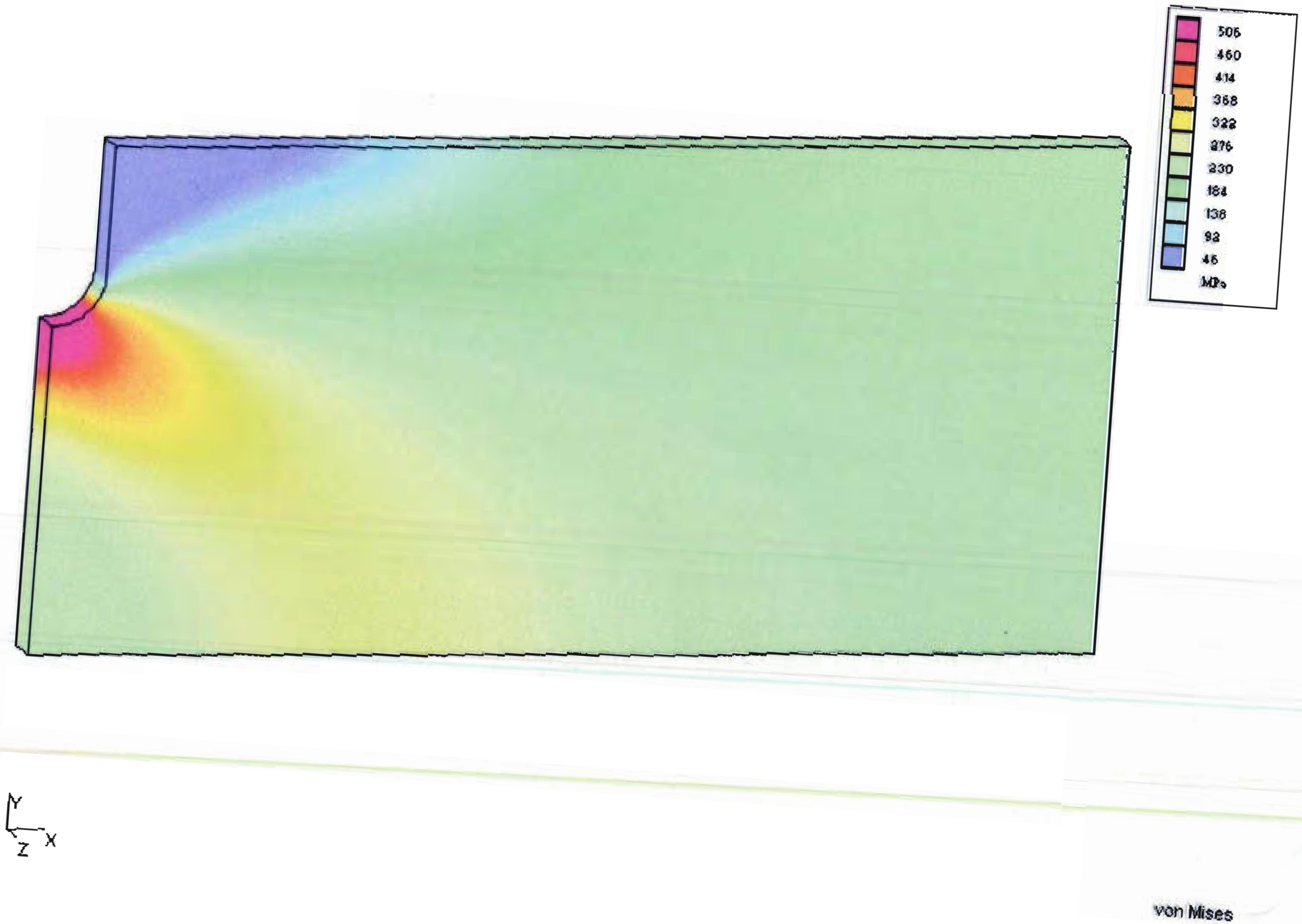
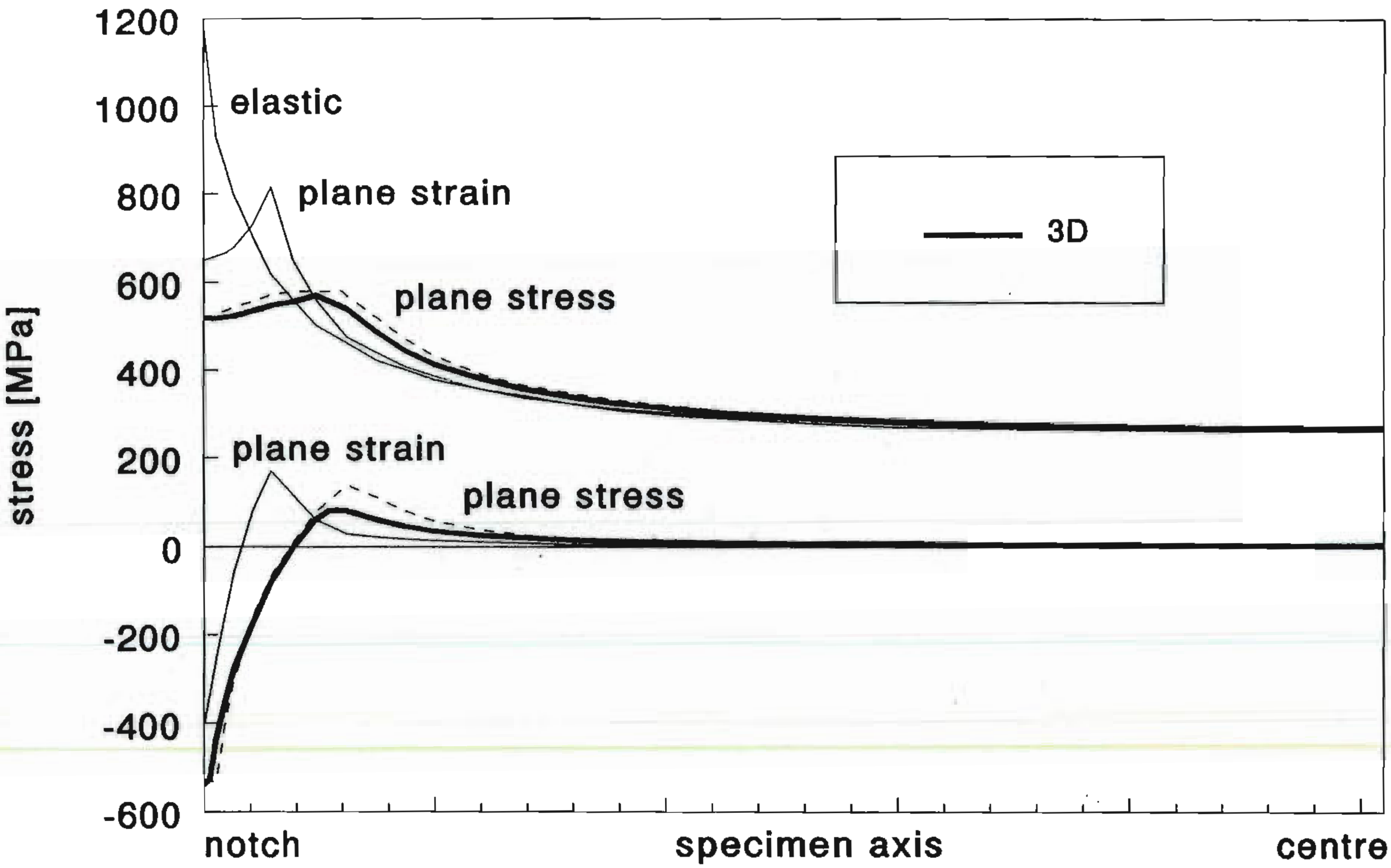
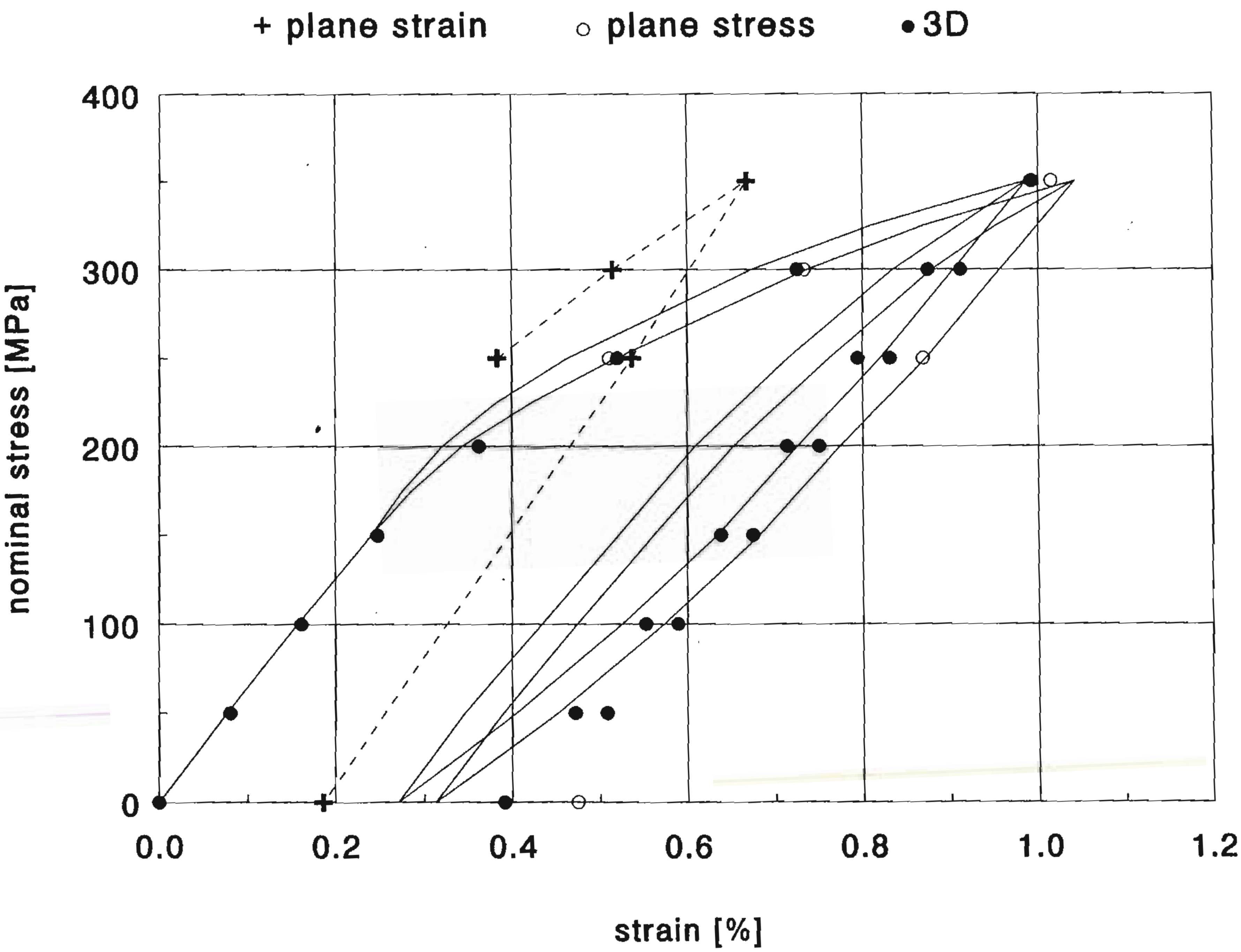
Material data: $\sigma_Y = 505\,\mathsf{MPa}$, $\sigma_D = \sigma_Y/1.5 = 337\,\mathsf{MPa}$

Loading: $\sigma^{\mathsf{nom}} = 350\,\mathsf{MPa} > 337\,\mathsf{MPa}$ (small scale cyclic plasticity expected)




506
460
414
368
322
276
230
184
138
92
46
MPa
Y
Z
X
von Mises

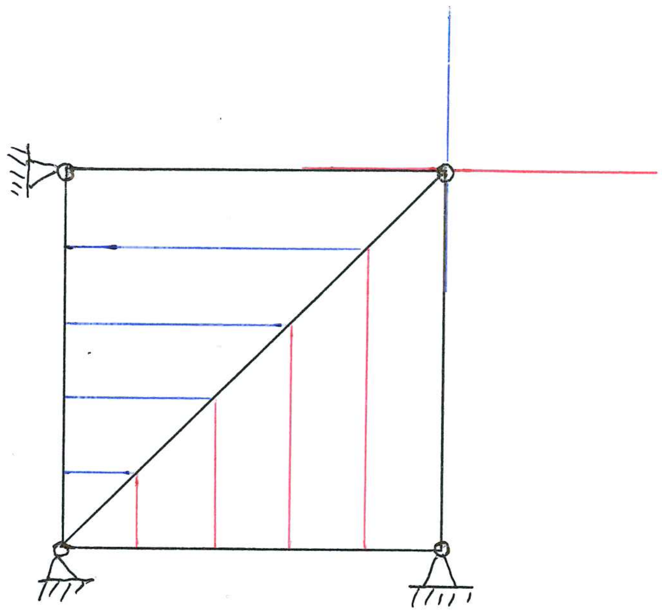

1200
1000
800
600
400
200
0
-200
-400
-600
stress [MPa]
elastic
plane strain
plane stress
plane strain
plane stress
3D
notch
specimen axis
centre



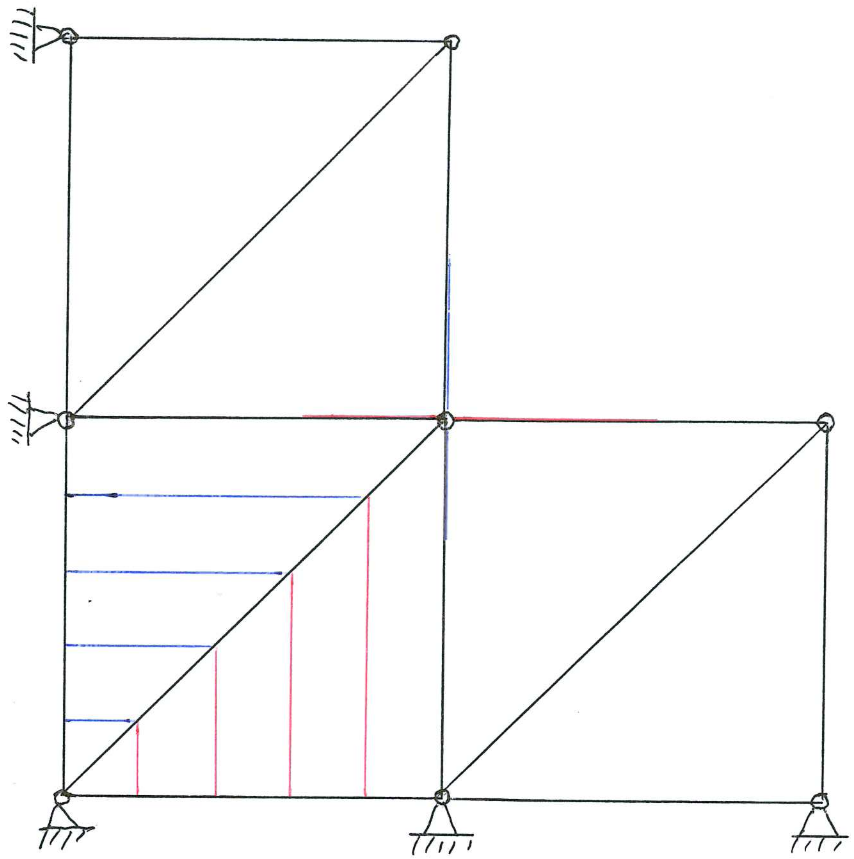
+ plane strain
○ plane stress
● 3D
nominal stress [MPa]
400
300
200
100
0
0.0
0.2
0.4
0.6
0.8
1.0
1.2
strain [%]

VOLUMETRIC LOCKING

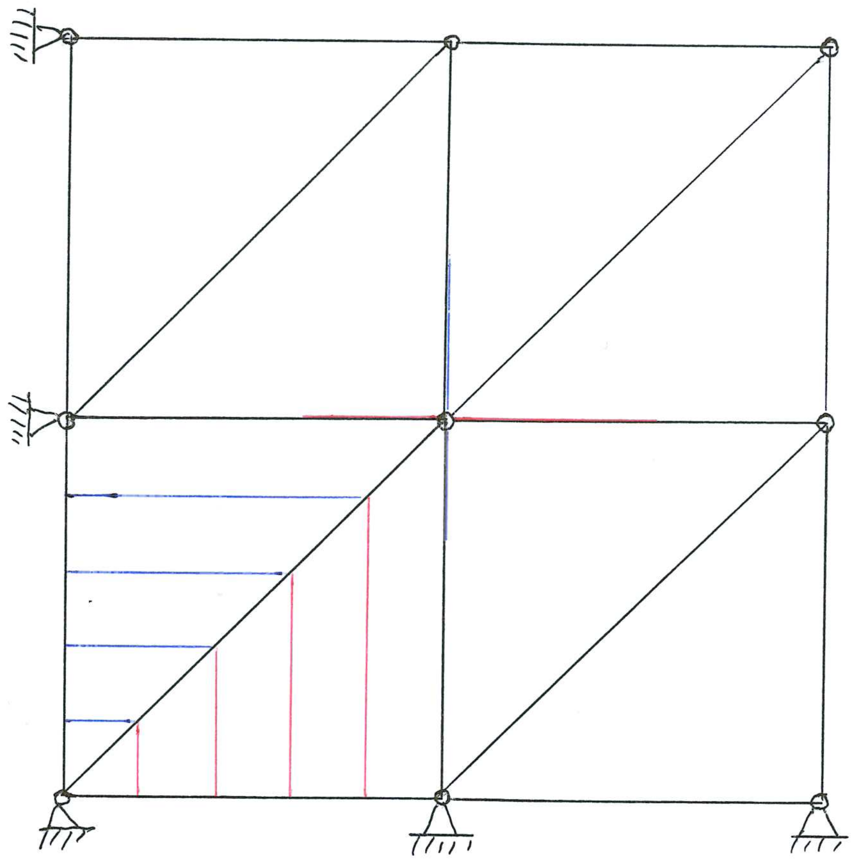
Volumetric locking

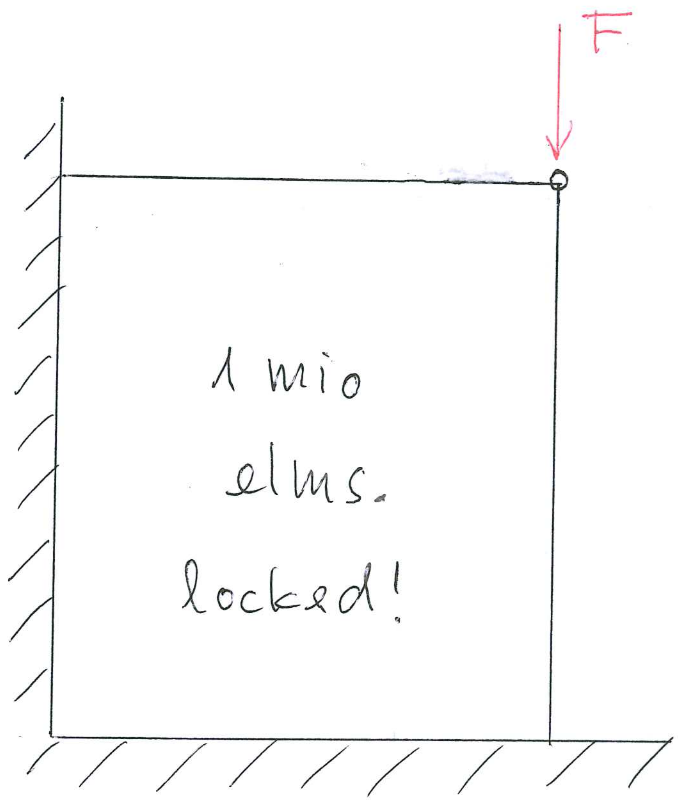


Volumetric locking

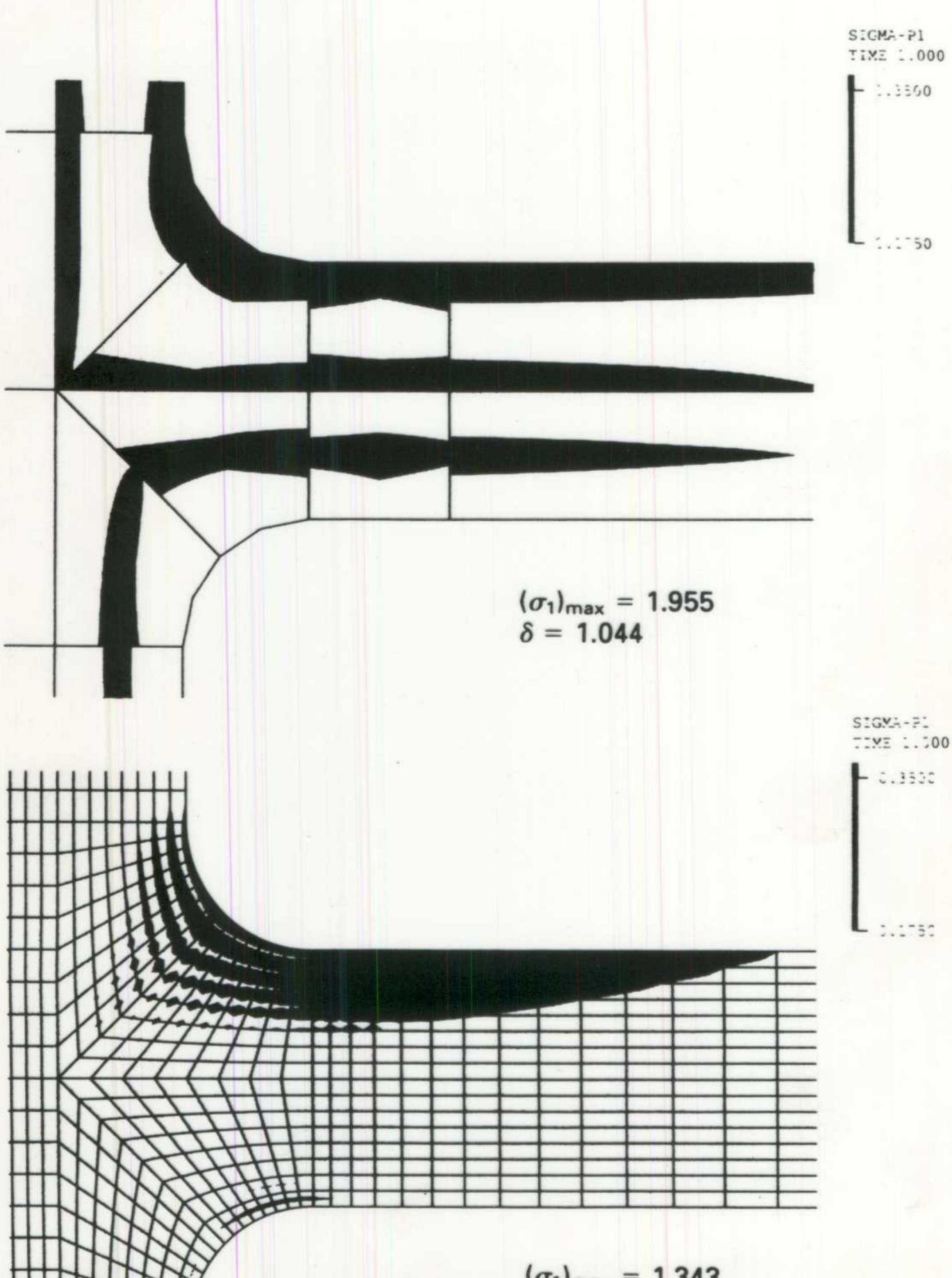


Volumetric locking



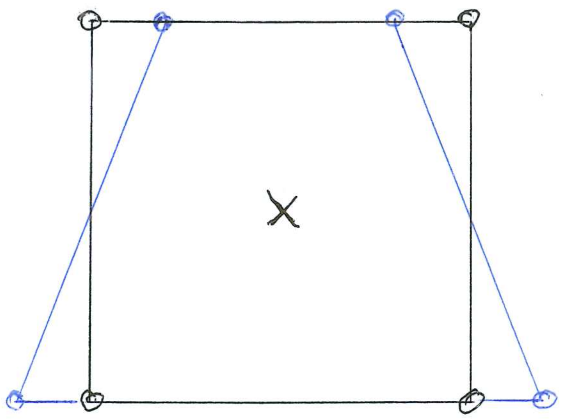


Convergence to a wrong limit!

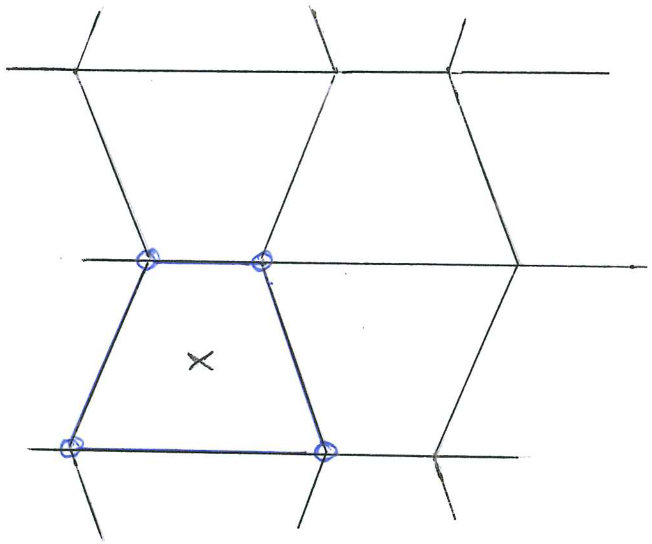


(c) Displacement-based element solution results for the case Poisson's ratio

Reduced integration



zero energy mode

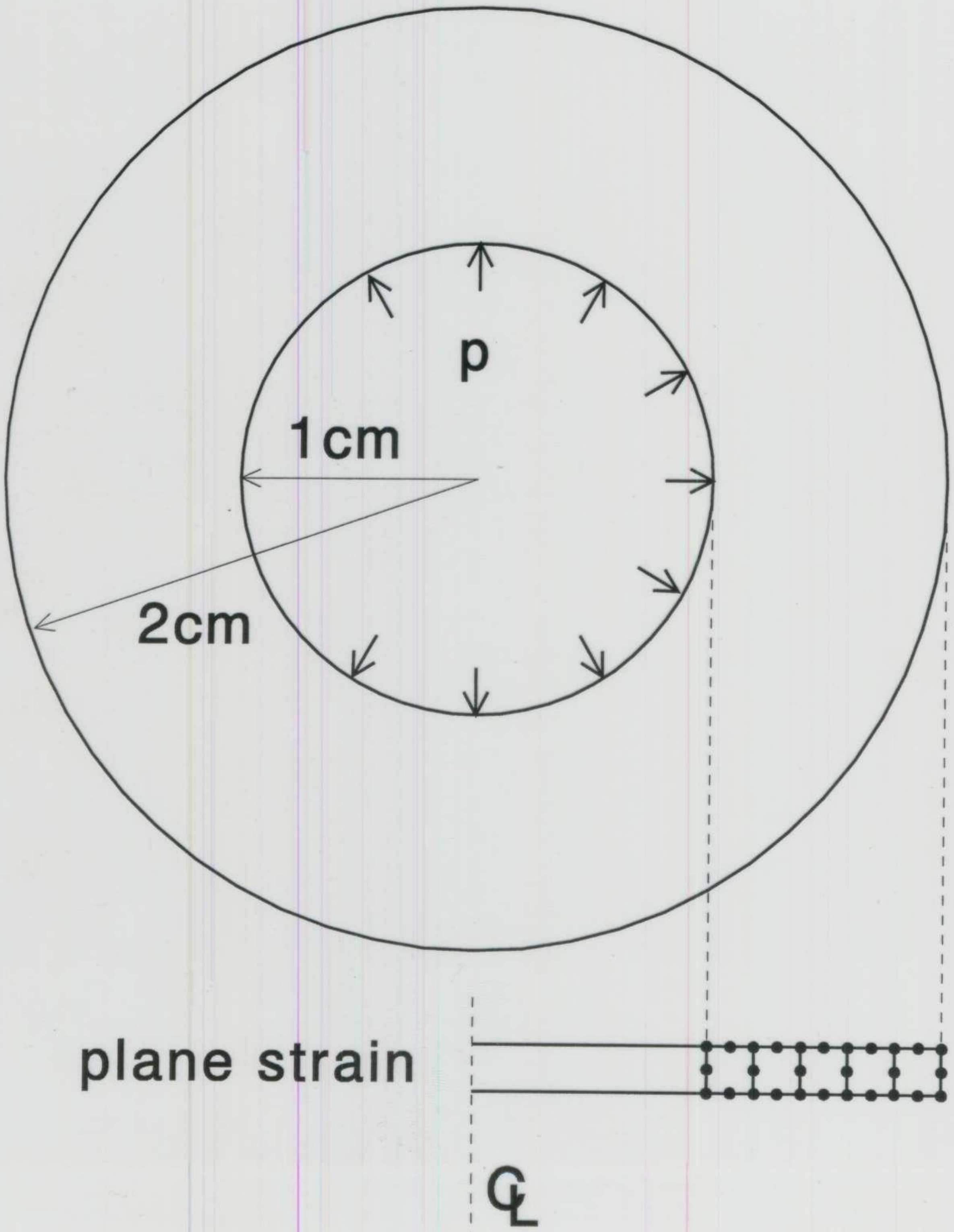


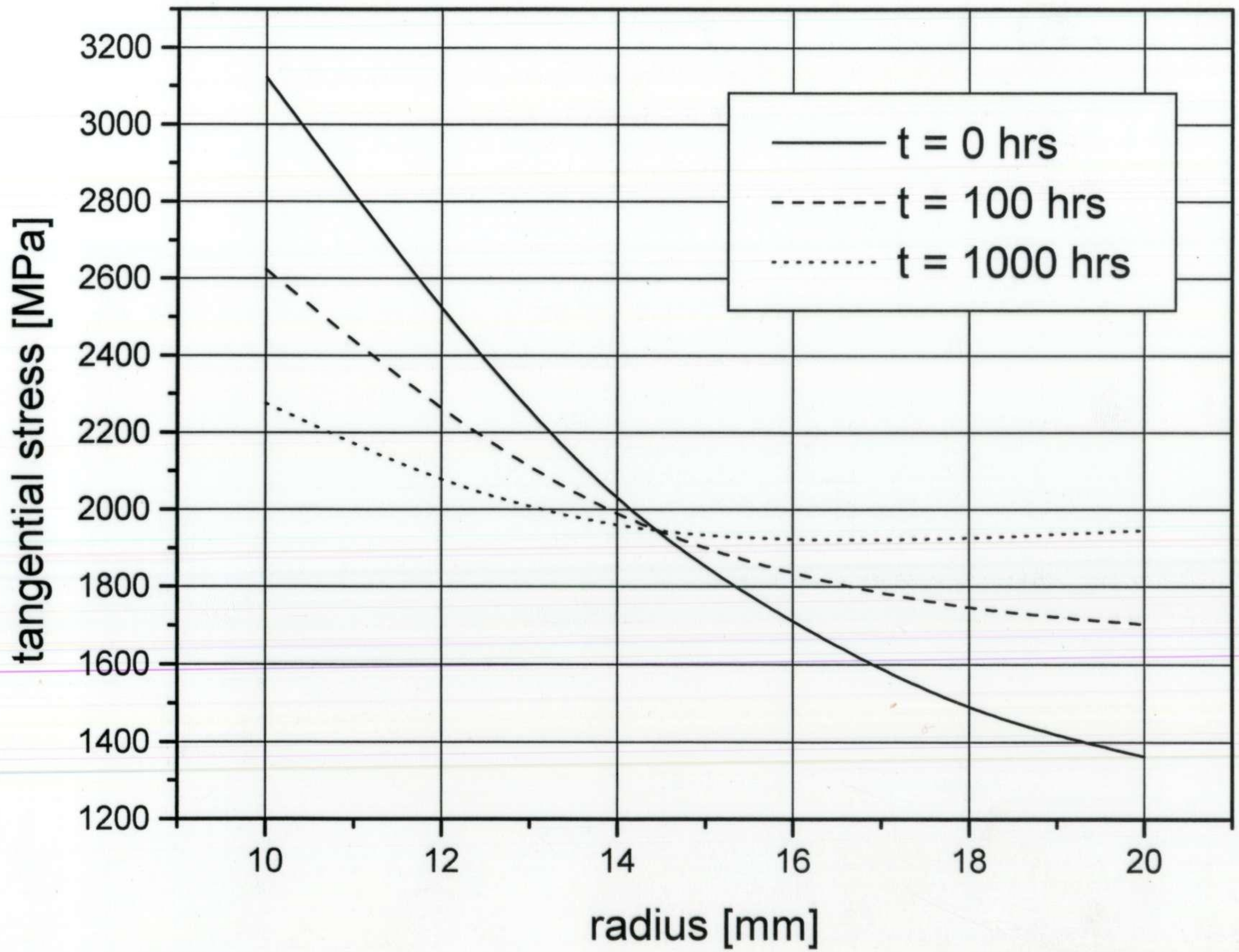
hourglassing

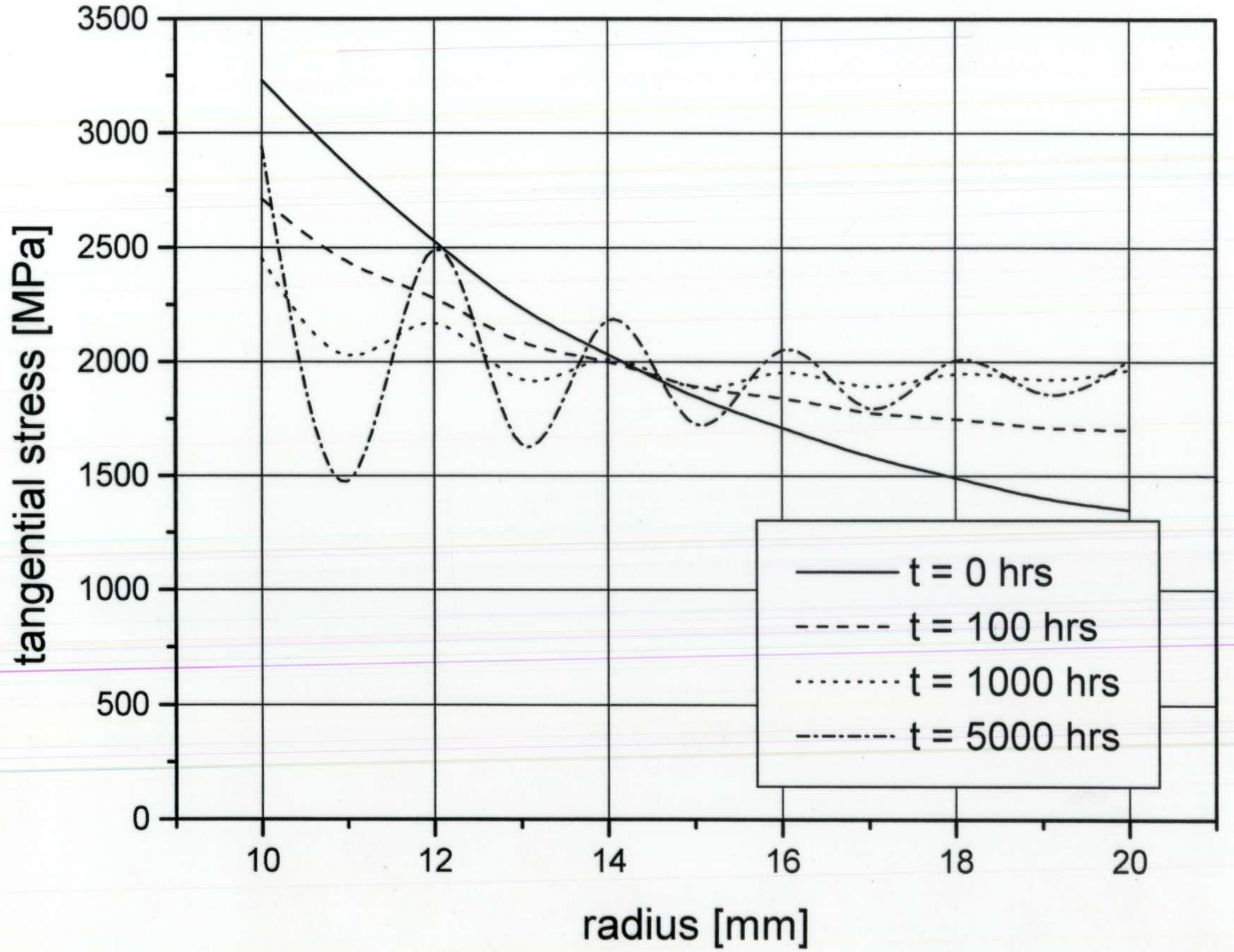
(A structruded mesh is especially prone to hourglassing.)

Integrating stiff systems

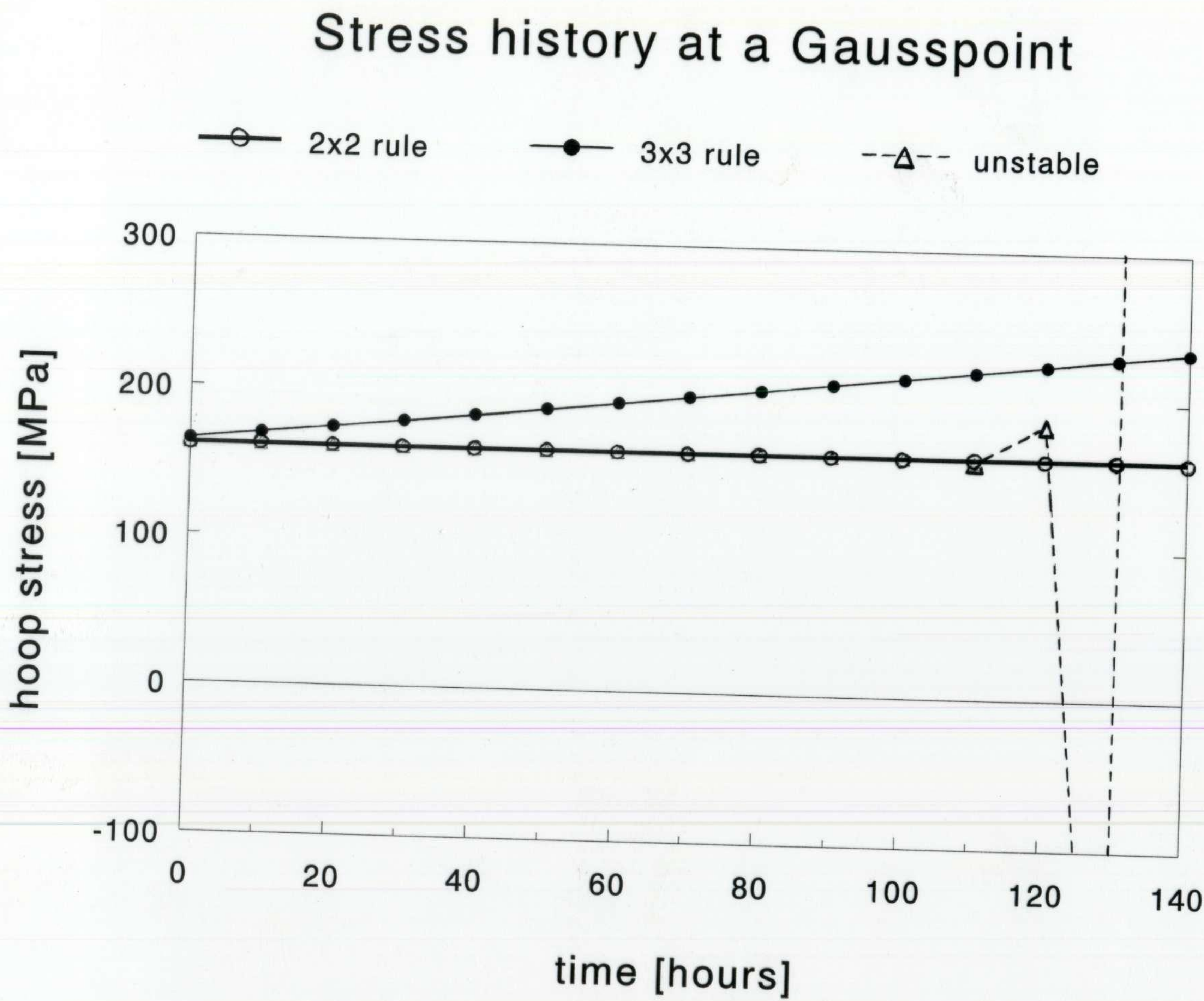
Plešek, J., Korouš, J.: Explicit integration method with time step control for viscoplasticity and creep. Adv. Engrg. Software, **33**, No. 7–10, pp. 621–630, 2002.


p
1cm
2cm
plane strain
CL


t = 0 hrs
t = 100 hrs
t = 1000 hrs
tangential stress [MPa]
3200
3000
2800
2600
2400
2200
2000
1800
1600
1400
1200
10
12
14
16
18
20
radius [mm]



3500
3000
2500
2000
1500
1000
500
0
tangential stress [MPa]
10
12
14
16
18
20
radius [mm]
t = 0 hrs
t = 100 hrs
t = 1000 hrs
t = 5000 hrs



Stress history at a Gausspoint
2x2 rule
3x3 rule
unstable
hoop stress [MPa]
300
200
100
0
-100
0
20
40
60
80
100
120
140
time [hours]

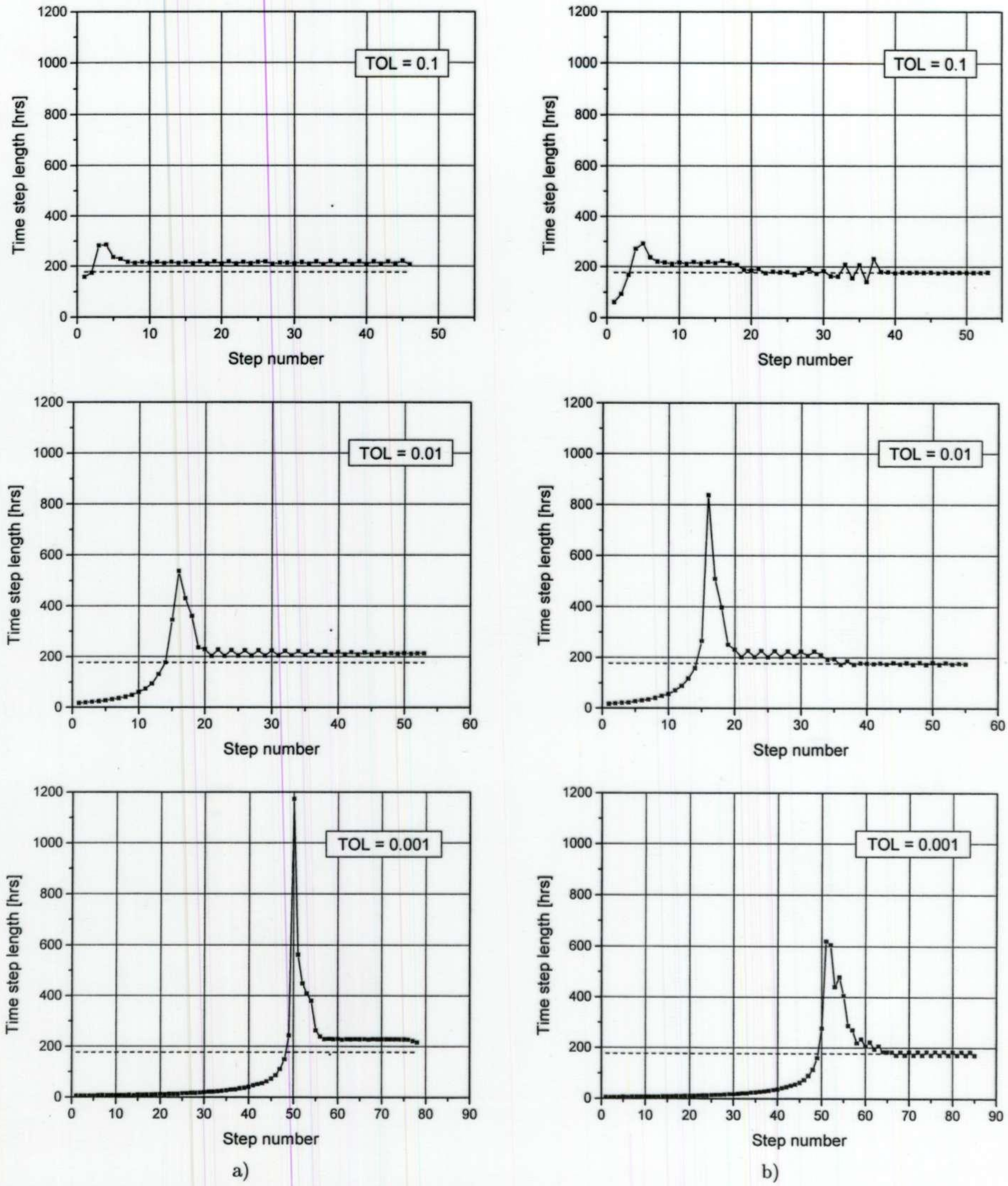


Figure 4: History of the optimised time step for a) 2×2 and b) 3×3 quadrature rules. Horizontal dashed line depicts the critical time step obtained from Cormeaus' formula ($\Delta t_{crit} = 177.33$ hrs).

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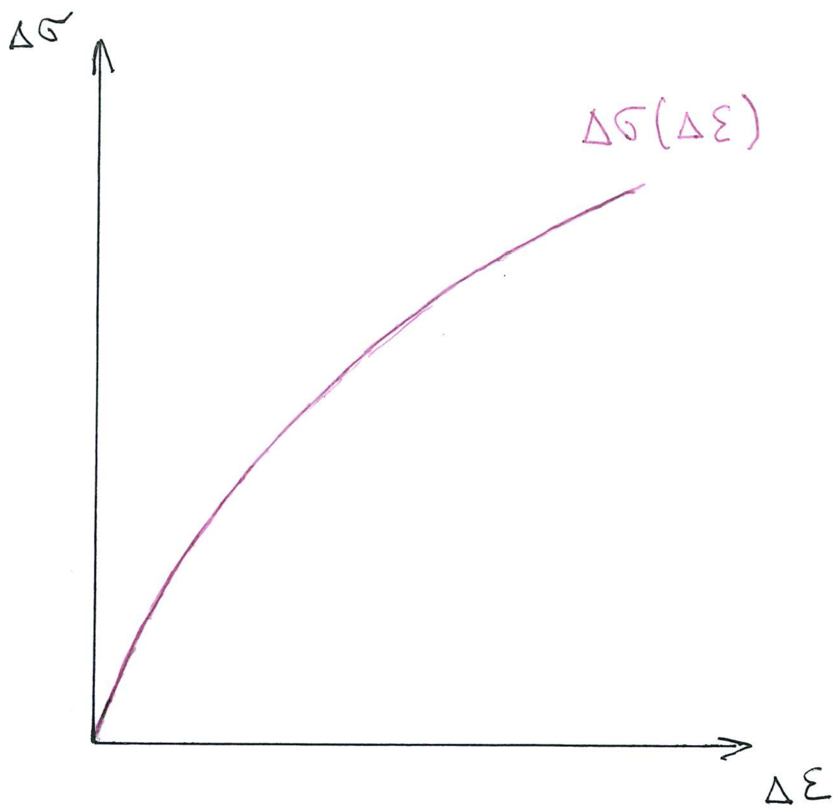
CONSISTENT TANGENT OPERATORS

Euler backward

In FEM: $\dot{\varepsilon} = \dfrac{\Delta \varepsilon}{\Delta t} = \text{const.}$

$$\boxed{\dot{\sigma} = C^{ep} \dot{\varepsilon}}$$



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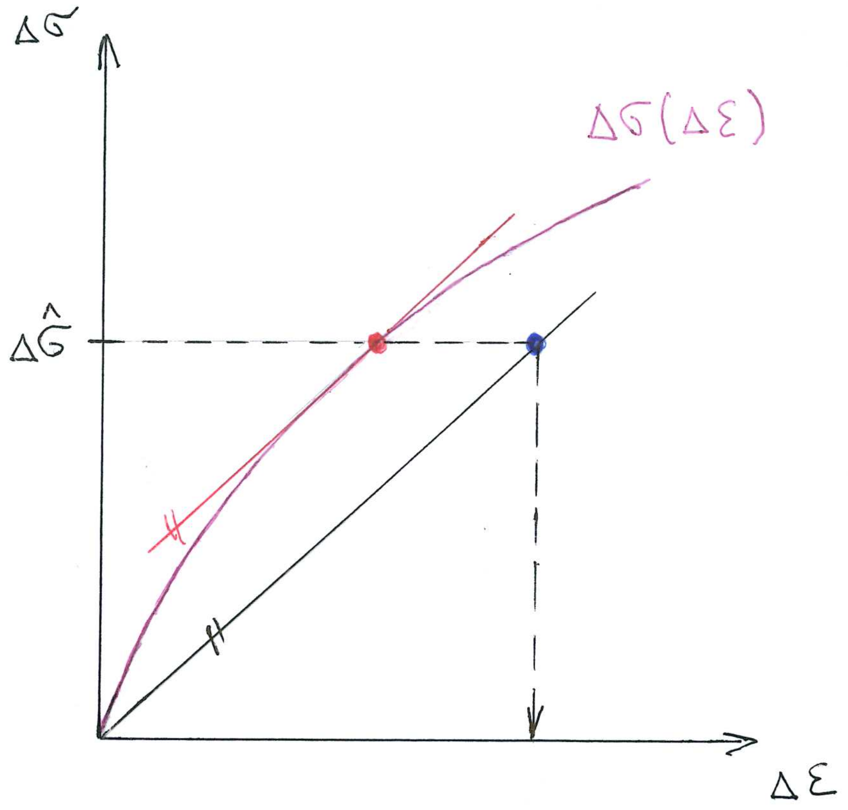
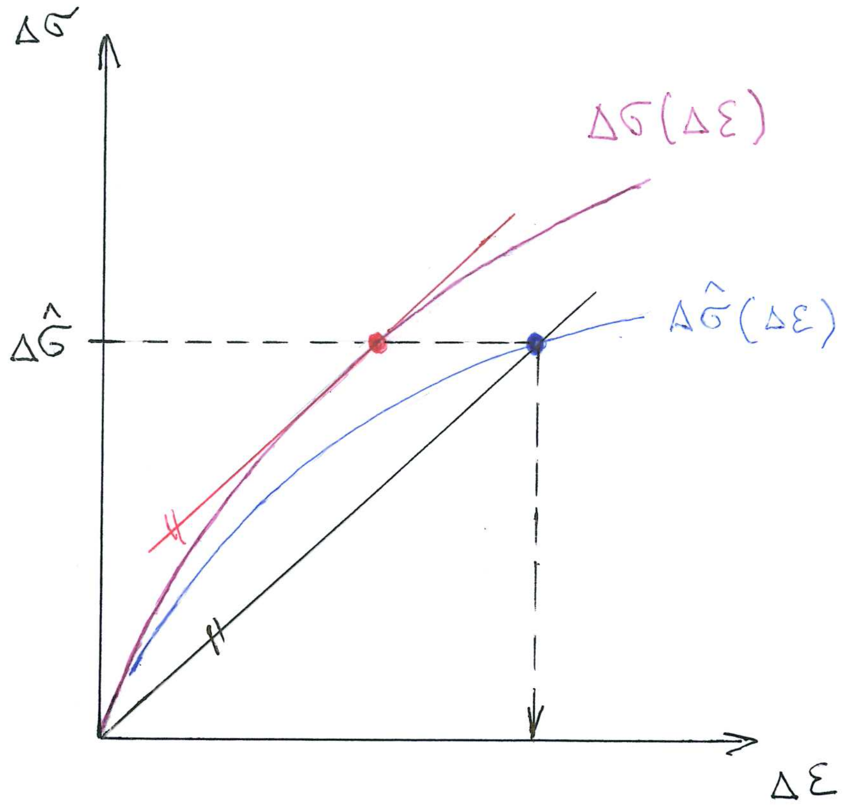


Euler backward



In FEM: $\dot{\varepsilon} = \dfrac{\Delta\varepsilon}{\Delta t} = \text{const.}$

$$\boxed{\dot{\sigma} = C^{ep}\dot{\varepsilon}}$$

Continuum tangent operator

$$d\sigma = C^{ep} d\varepsilon$$

Consistent tangent operator

$$d\hat{\sigma} = H^{ep} d\varepsilon$$

(used by the Newton–Raphson)

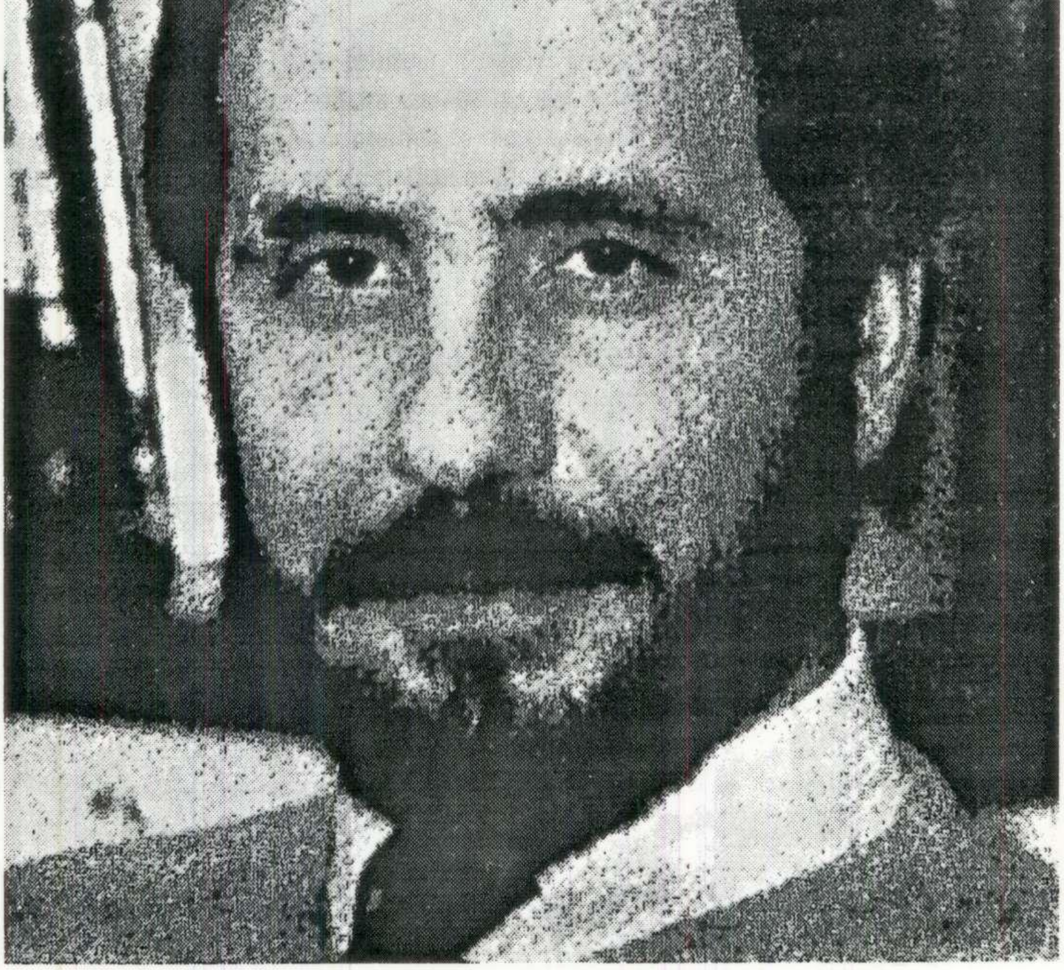
INTERDISCIPLINARY APPLIED MATHEMATICS

MECHANICS AND MATERIALS

Computational Inelasticity

J.C. Simo
T.J.R. Hughes

Springer



JUAN CARLOS SIMO

1952-1994