Institute of Thermomechanics ASCR

LAGRANGE DESCRIPTION III
EULER DESCRIPTION I

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Contents

Rotation

Special case

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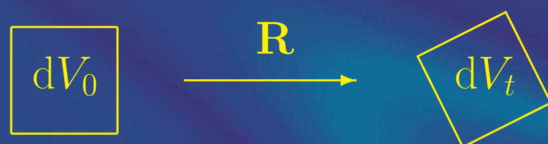

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$$\boxed{\mathrm{d}V_0} \xrightarrow{\mathbf{U}} \boxed{\mathrm{d}V_t}$$

Composition

Stretch and rotation

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$$\mathbf{F} = \mathbf{F}_N \mathbf{F}_{N-1} \cdots \mathbf{F}_2 \mathbf{F}_1$$

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It follows that $[U]^2 = [\Phi][\Lambda][\Phi]^T[\Phi][\Lambda][\Phi]^T = [\Phi][\Lambda]^2[\Phi]^T = [C]$.

Polar decomposition

Theorem

$$\boxed{\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}} \qquad \text{uniquely}$$

The right and left stretch tensors

$\mathbf{U}\,,\ \mathbf{V}$ sym+def, having the same principal stretches $\lambda_k > 0$

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. . . verifies as

$$\det|F| = \underbrace{\det|R|}_{\pm 1}\,\underbrace{\det|U|}_{>0} = J > 0$$

Two families of strain tensors

Grand scheme

$$
\begin{tikzcd}
\mathrm{d}V_m \arrow[r, "\mathbf{V}"] & \mathrm{d}V_t \\
\mathrm{d}V_0 \arrow[u, "\mathbf{R}"] \arrow[r, "\mathbf{U}"] & \mathrm{d}V_m \arrow[u, "\mathbf{R}"']
\end{tikzcd}
$$

Lagrange family

$$\mathbf{E} = f(\mathbf{U}) \quad \text{(e.g. Green-Lagrange)}$$

Euler family

$$\mathbf{A} = f(\mathbf{V}) \quad \text{(e.g. Euler-Almansi)}$$

Examples of Lagrangean tensors

Green-Lagrange

$$\mathbf{e} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}) = \frac{1}{2}(\mathbf{U}^T\mathbf{R}^T\mathbf{R}\mathbf{U} - \mathbf{I}) = \frac{1}{2}(\mathbf{U}^2 - \mathbf{I})$$

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$$\boldsymbol{\epsilon} = \frac{1}{2}(\mathbf{z} + \mathbf{z}^T) = \frac{1}{2}(\mathbf{F} + \mathbf{F}^T) - \mathbf{I} = \frac{1}{2}(\mathbf{R}\mathbf{U} + \mathbf{U}\mathbf{R}^T) - \mathbf{I}$$

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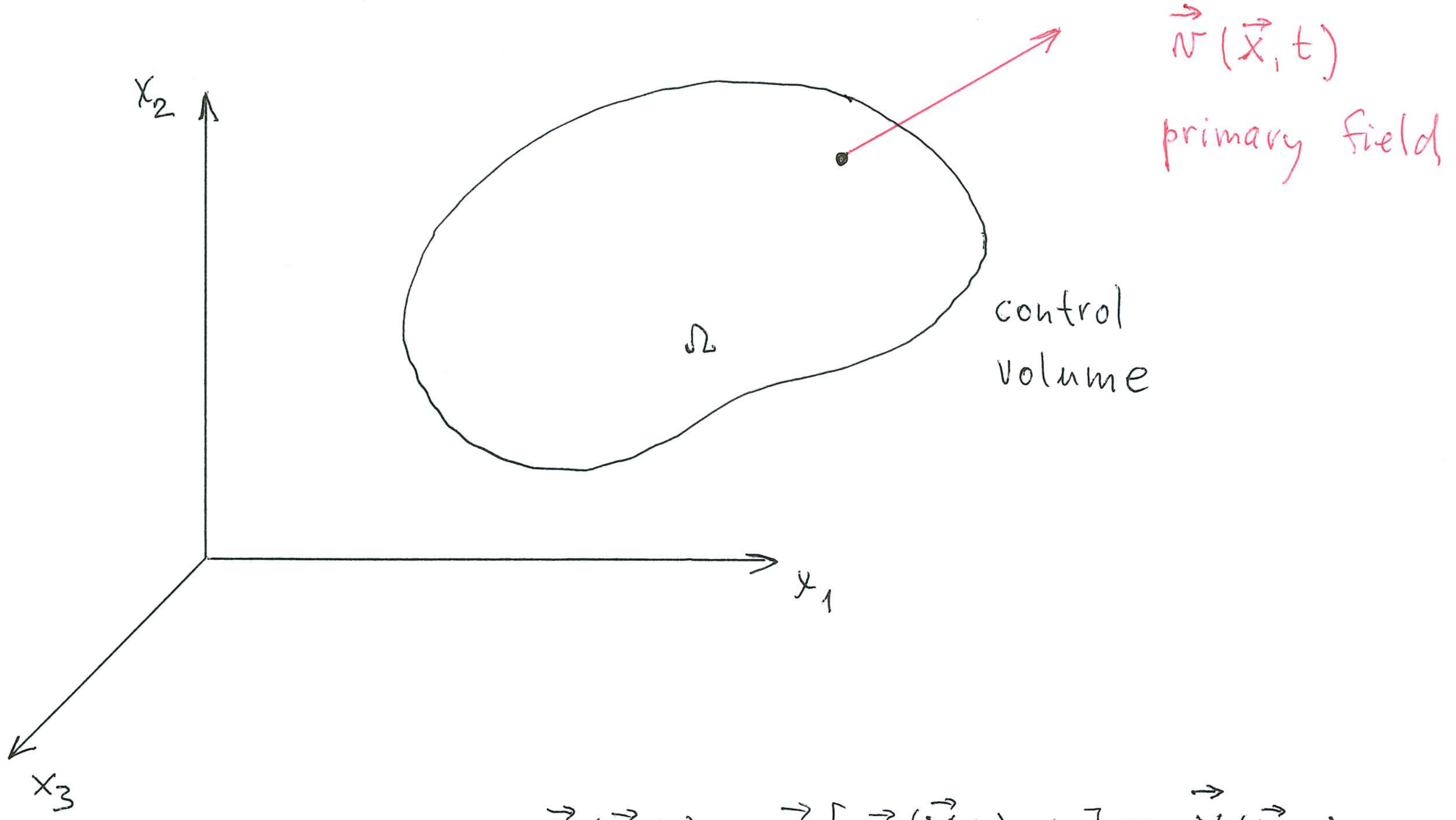
Biot

$$\mathbf{b} = \mathbf{U} - \mathbf{I} \qquad \text{for any } \mathbf{R}$$

EULER DESCRIPTION



$$\vec{v}(\vec{x},t) = \underbrace{\vec{v}\left[\vec{x}(\vec{X},t),t\right] \equiv \vec{V}(\vec{X},t)}_{\text{identical functions}}$$

Composed mappings

Composition of a mapping

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Quiz

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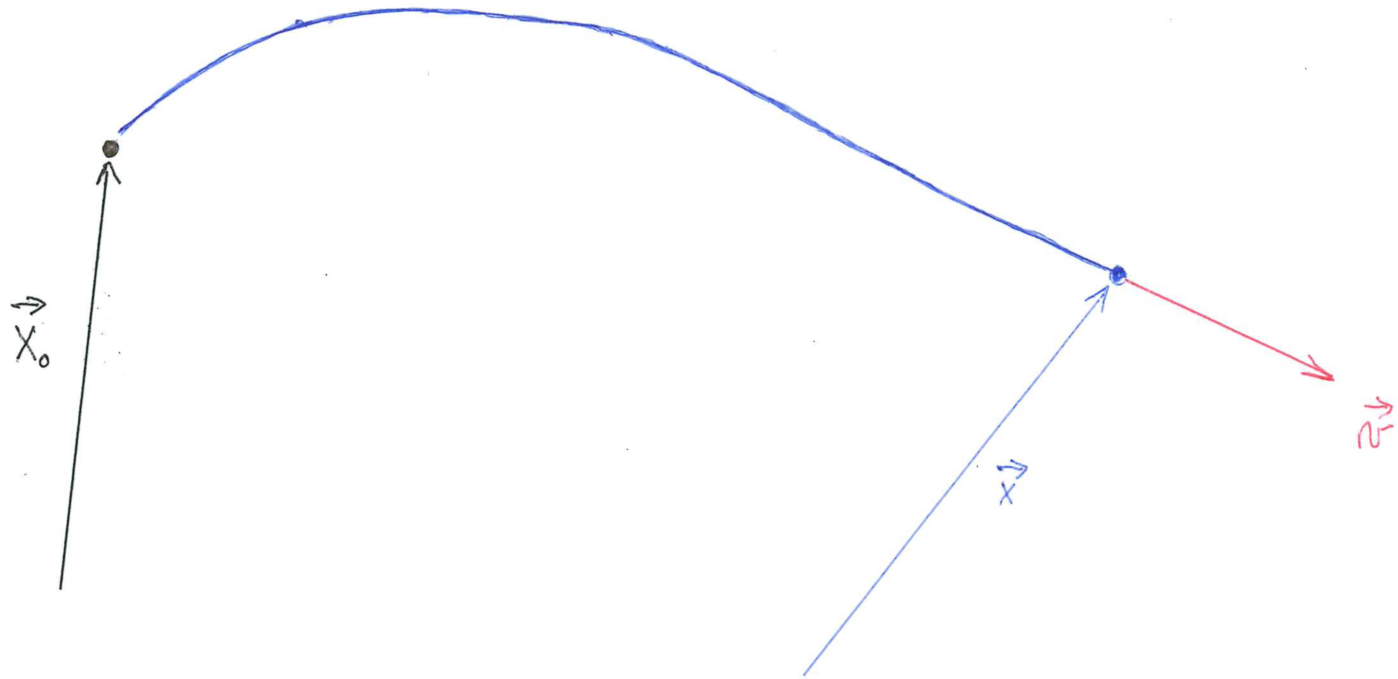
trajectory of a particle



trajectory: $\boxed{\vec{x}(\vec{X}_0, t)}$ velocity: $\vec{v} = \frac{d}{dt}\vec{x}(\vec{X}_0, t)$

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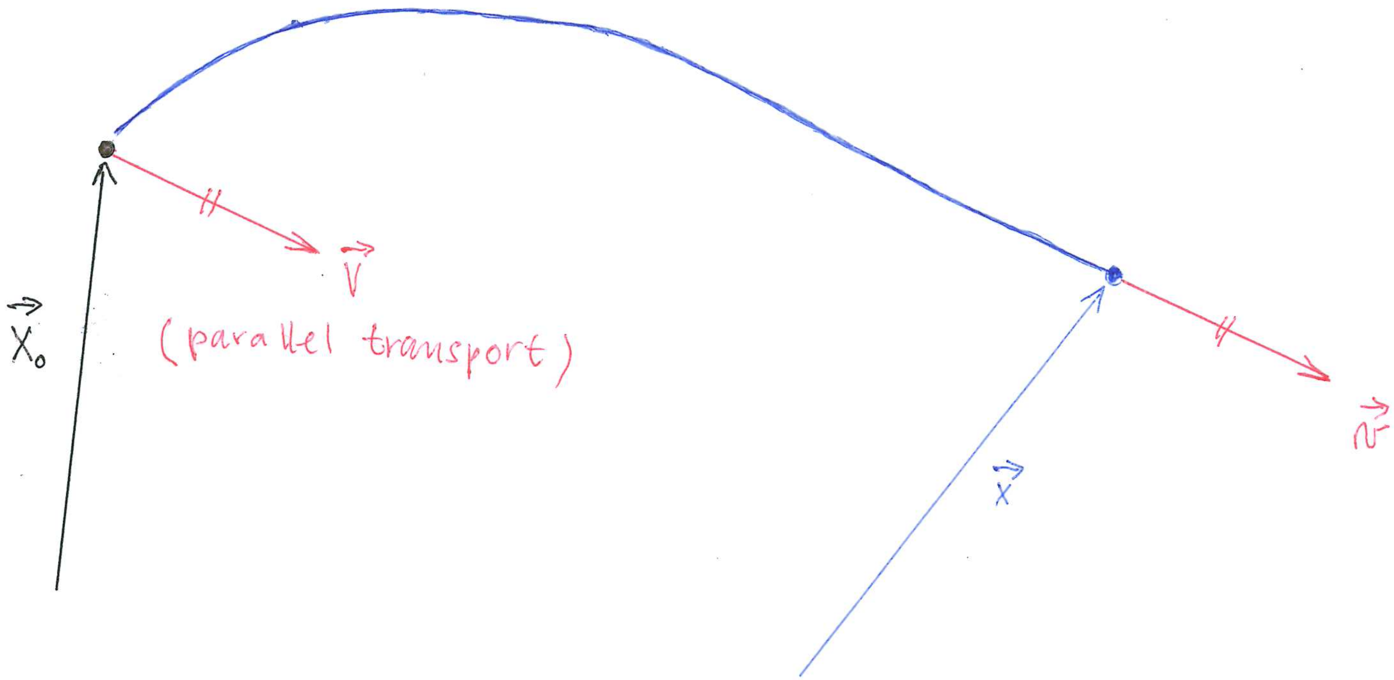

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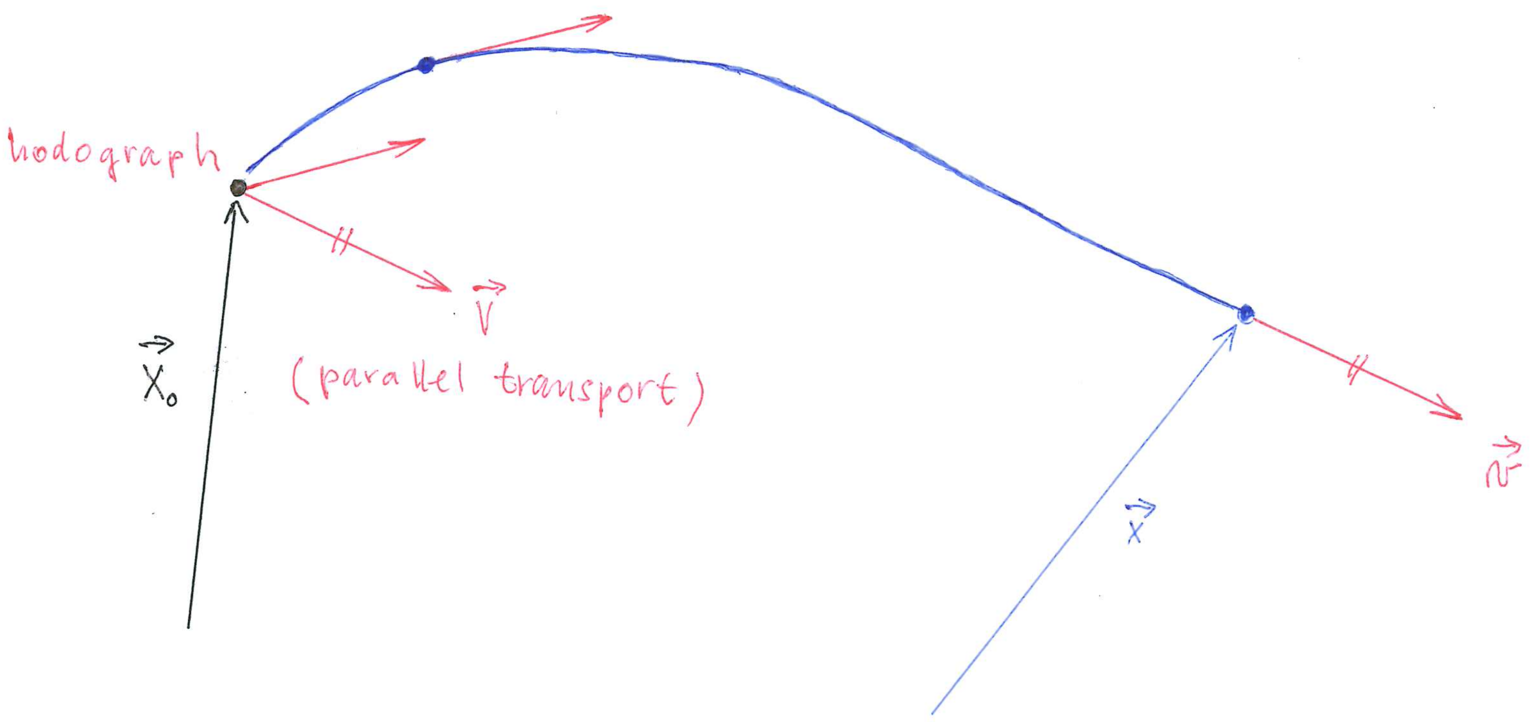

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Acceleration

Definition

$$\boxed{\mathbf{A}(\mathbf{X},t) = \frac{\partial \mathbf{V}}{\partial t} \quad \mapsto \quad \mathbf{a}(\mathbf{x},t)}$$

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No L-field needed!

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$$\boxed{\mathbf{a} = \frac{\partial \mathbf{v}}{\partial t} + \frac{\partial \mathbf{v}}{\partial x_j} v_j}$$

Example 1: Domain mapping (1/2)

Mapping

$$
\begin{aligned}
x_1 &= X_1 + at^2 \\
x_2 &= X_2 + bX_2t + ct^2
\end{aligned}
$$

Inverse

$$
\begin{aligned}
X_1 &= x_1 - at^2 \\
X_2 &= \frac{x_2 - ct^2}{1 + bt}
\end{aligned}
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Lagrange description

$$\begin{aligned} V_1(X_1, X_2, t) &= 2at \\ V_2(X_1, X_2, t) &= bX_2 + 2ct \end{aligned}$$

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Numerical example: $X_1 = X_2 = 0$, $t = 3$

Example 1: Domain mapping (1/2)

Mapping

$$x_1 = X_1 + at^2$$
$$x_2 = X_2 + bX_2t + ct^2$$

Inverse

$$X_1 = x_1 - at^2$$
$$X_2 = \frac{x_2 - ct^2}{1 + bt}$$

Lagrange description

$$V_1(0,0,3) = 6a$$
$$V_2(X_1, X_2, t) = bX_2 + 2ct$$

Euler description

$$v_1(x_1, x_2, t) = 2at$$
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Euler description

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Numerical example: $X_1 = X_2 = 0$, $t = 3$

Example 1: Domain mapping (1/2)

Mapping

$$\begin{aligned} x_1 &= 9a \\ x_2 &= X_2 + bX_2t + ct^2 \end{aligned}$$

Inverse

$$\begin{aligned} X_1 &= x_1 - at^2 \\ X_2 &= \frac{x_2 - ct^2}{1 + bt} \end{aligned}$$

Lagrange description

$$\begin{aligned} V_1(0,0,3) &= 6a \\ V_2(0,0,3) &= 6c \end{aligned}$$

Euler description

$$\begin{aligned} v_1(x_1,x_2,t) &= 2at \\ v_2(x_1,x_2,t) &= b\frac{x_2 - ct^2}{1 + bt} + 2ct \end{aligned}$$

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Numerical example: $X_1 = X_2 = 0$, $t = 3$

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Lagrange description

$$\begin{aligned} V_1(0,0,3) &= 6a \\ V_2(0,0,3) &= 6c \end{aligned}$$

Euler description

$$\begin{aligned} v_1(9a,9c,3) &= 6a \\ v_2(9a,9c,3) &= b\frac{9c - 9c}{1 + 3b} + 6c \end{aligned}$$

Numerical example: $X_1 = X_2 = 0$, $t = 3$

Example 1: Domain mapping (1/2)

Mapping

$$x_1 = 9a$$
$$x_2 = 9c$$

Inverse

$$X_1 = 0$$
$$X_2 = 0$$

Lagrange description

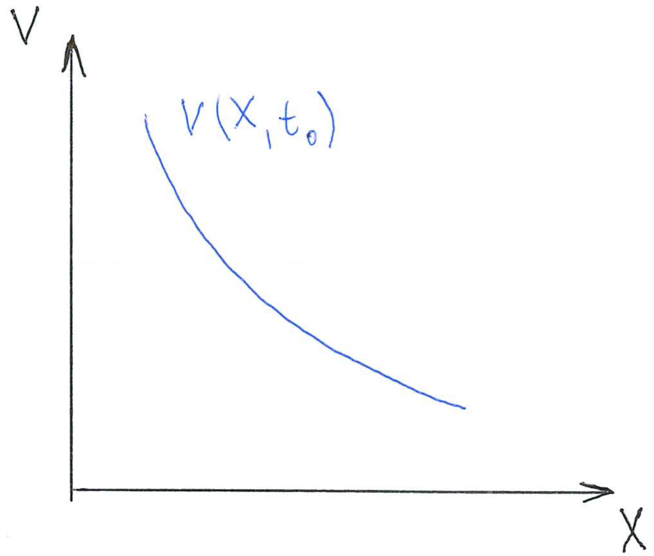
$$V_1(0, 0, 3) = 6a$$
$$V_2(0, 0, 3) = 6c$$

Euler description

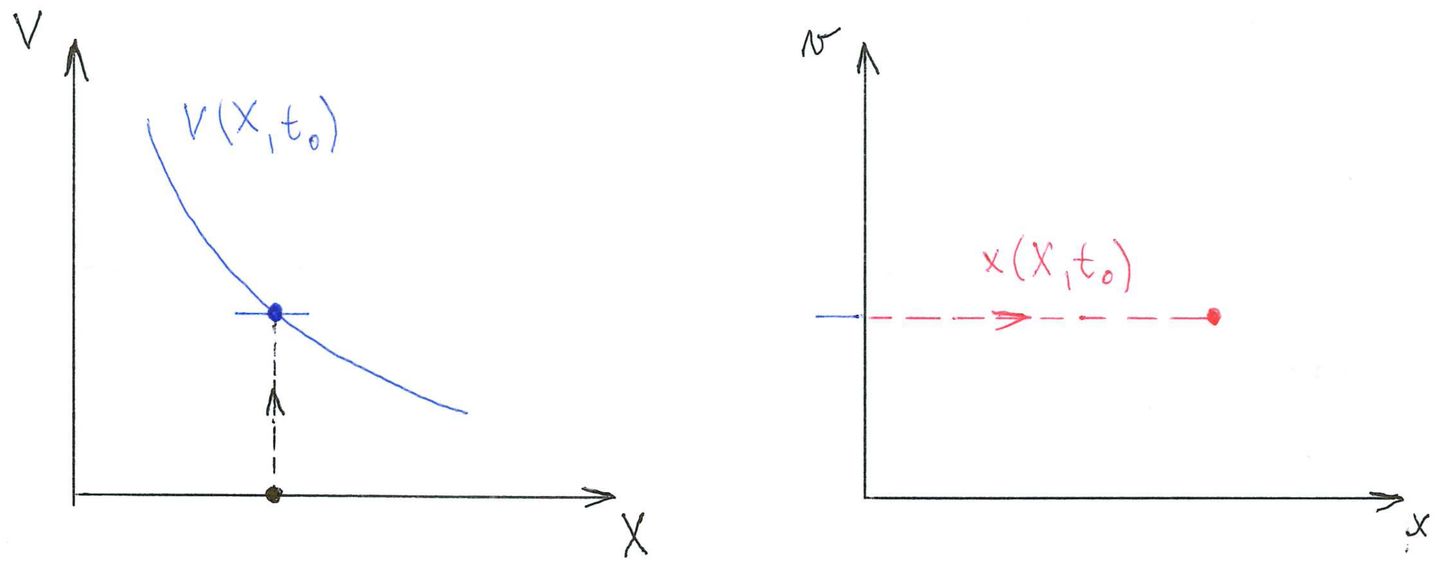
$$v_1(9a, 9c, 3) = 6a$$
$$v_2(9a, 9c, 3) = 6c$$

Numerical example: $X_1 = X_2 = 0$, $t = 3$

For a fixed t_0:



For a fixed t_0:



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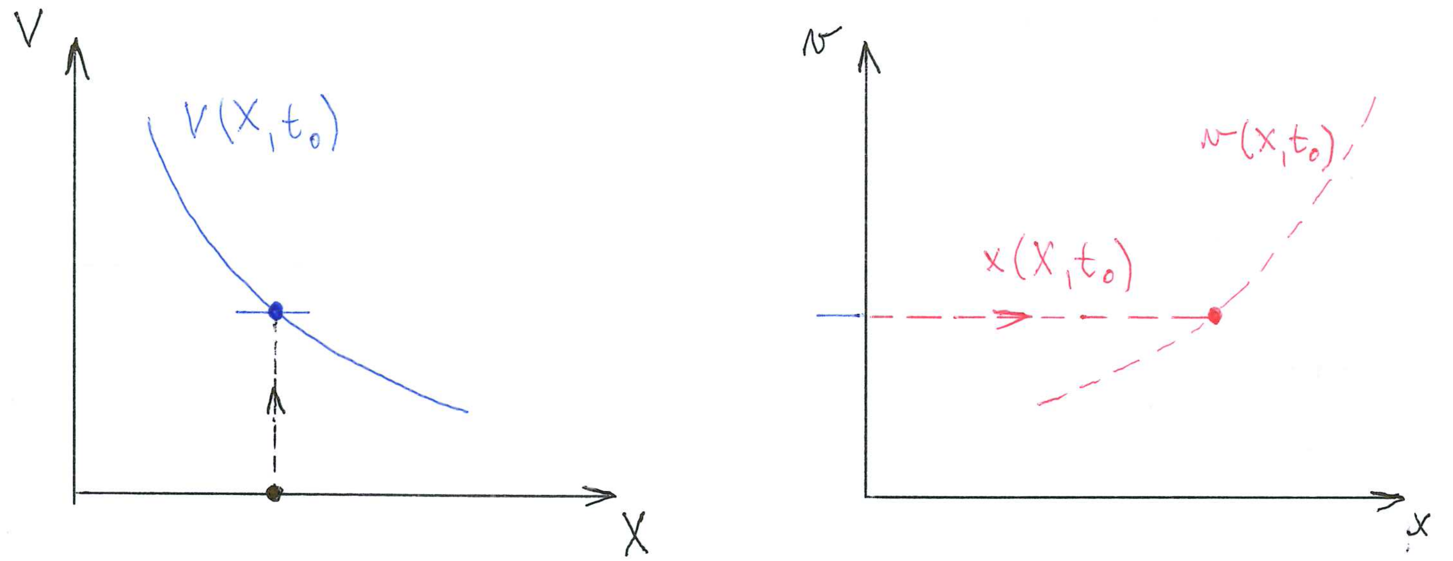

Same values, different functions, different derivatives!

Example 1: Domain mapping (2/2)

$$
\begin{aligned}
V_1(X_1, X_2, t) &= 2at \\
V_2(X_1, X_2, t) &= bX_2 + 2ct
\end{aligned}
$$

Example 1: Domain mapping (2/2)

$$
\begin{aligned}
V_1(X_1, X_2, t) &= 2at & A_1(X_1, X_2, t) &= 2a = a_1(x_1, x_2, t) \\
V_2(X_1, X_2, t) &= bX_2 + 2ct & A_2(X_1, X_2, t) &= 2c = a_2(x_1, x_2, t)
\end{aligned}
$$

Example 1: Domain mapping (2/2)

stored result: $A_1 = a_1 = 2a$, $A_2 = a_2 = 2c$

Euler field

$$
\begin{aligned}
v_1(x_1, x_2, t) &= 2at \\
v_2(x_1, x_2, t) &= b\frac{x_2 - ct^2}{1 + bt} + 2ct
\end{aligned}
$$

Example 1: Domain mapping (2/2)

stored result: $A_1 = a_1 = 2a,\ A_2 = a_2 = 2c$

Euler field

$$v_1(x_1, x_2, t) = 2at \qquad a_1(x_1, x_2, t) = \frac{\partial v_1}{\partial t} + \frac{\partial v_1}{\partial x_j} v_j$$

$$v_2(x_1, x_2, t) = b\frac{x_2 - ct^2}{1 + bt} + 2ct \qquad a_2(x_1, x_2, t) = \frac{\partial v_2}{\partial t} + \frac{\partial v_2}{\partial x_j} v_j$$

Example 1: Domain mapping (2/2)

stored result: $A_1 = a_1 = 2a$ (checks), $A_2 = a_2 = 2c$

Euler field

$$v_1(x_1, x_2, t) = 2at \qquad a_1(x_1, x_2, t) = \frac{\partial v_1}{\partial t} + \frac{\partial v_1}{\partial x_j} v_j = 2a$$

$$v_2(x_1, x_2, t) = b\frac{x_2 - ct^2}{1 + bt} + 2ct \qquad a_2(x_1, x_2, t) = \frac{\partial v_2}{\partial t} + \frac{\partial v_2}{\partial x_j} v_j$$

Example 1: Domain mapping (2/2)

stored result: $A_1 = a_1 = 2a$ (checks), $A_2 = a_2 = 2c$

Euler field

$$v_1(x_1, x_2, t) = 2at \qquad a_1(x_1, x_2, t) = \frac{\partial v_1}{\partial t} + \frac{\partial v_1}{\partial x_j} v_j = 2a$$

$$v_2(x_1, x_2, t) = b\frac{x_2 - ct^2}{1 + bt} + 2ct \qquad a_2(x_1, x_2, t) = \frac{\partial v_2}{\partial t} + \frac{\partial v_2}{\partial x_j} v_j$$

$$\frac{\partial v_2}{\partial t} =$$

$$+$$

$$\frac{\partial v_2}{\partial x_j} v_j =$$

Example 1: Domain mapping (2/2)

stored result: $A_1 = a_1 = 2a$ (checks), $A_2 = a_2 = 2c$

Euler field

$$v_1(x_1, x_2, t) = 2at \qquad a_1(x_1, x_2, t) = \frac{\partial v_1}{\partial t} + \frac{\partial v_1}{\partial x_j} v_j = 2a$$

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$$\frac{\partial v_2}{\partial t} = b\frac{-2ct(1 + bt) - (x_2 - ct^2)b}{(1 + bt)^2} + 2c$$

$$+$$

$$\frac{\partial v_2}{\partial x_j} v_j =$$

Example 1: Domain mapping (2/2)

stored result: $A_1 = a_1 = 2a$ (checks), $A_2 = a_2 = 2c$

Euler field

$$v_1(x_1,x_2,t) = 2at \qquad a_1(x_1,x_2,t) = \frac{\partial v_1}{\partial t} + \frac{\partial v_1}{\partial x_j} v_j = 2a$$

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$$+$$

$$\frac{\partial v_2}{\partial x_j} v_j = \frac{b}{1+bt}\left(b\frac{x_2 - ct^2}{1+bt} + 2ct\right)$$

Example 1: Domain mapping (2/2)

stored result: $A_1 = a_1 = 2a$ (checks), $A_2 = a_2 = 2c$

Euler field

$$
\begin{aligned}
v_1(x_1, x_2, t) &= 2at \\
v_2(x_1, x_2, t) &= b\frac{x_2 - ct^2}{1 + bt} + 2ct
\end{aligned}
\qquad
\begin{aligned}
a_1(x_1, x_2, t) &= \frac{\partial v_1}{\partial t} + \frac{\partial v_1}{\partial x_j} v_j = 2a \\
a_2(x_1, x_2, t) &= \frac{\partial v_2}{\partial t} + \frac{\partial v_2}{\partial x_j} v_j
\end{aligned}
$$

$$
\left.
\begin{aligned}
\frac{\partial v_2}{\partial t} &= b\frac{-2ct(1 + bt) - (x_2 - ct^2)b}{(1 + bt)^2} + 2c \\
+ & \\
\frac{\partial v_2}{\partial x_j} v_j &= \frac{b}{1 + bt}\left(b\frac{x_2 - ct^2}{1 + bt} + 2ct\right)
\end{aligned}
\right\} = 2c
$$

Example 1: Domain mapping (2/2)

stored result: $A_1 = a_1 = 2a$ (checks), $A_2 = a_2 = 2c$ (checks)

Euler field

$$v_1(x_1, x_2, t) = 2at \qquad a_1(x_1, x_2, t) = \frac{\partial v_1}{\partial t} + \frac{\partial v_1}{\partial x_j} v_j = 2a$$

$$v_2(x_1, x_2, t) = b\frac{x_2 - ct^2}{1 + bt} + 2ct \qquad a_2(x_1, x_2, t) = \frac{\partial v_2}{\partial t} + \frac{\partial v_2}{\partial x_j} v_j = 2c$$

$$\left.\begin{aligned} \frac{\partial v_2}{\partial t} &= b\frac{-2ct(1+bt) - (x_2 - ct^2)b}{(1+bt)^2} + 2c \\ &+ \\ \frac{\partial v_2}{\partial x_j} v_j &= \frac{b}{1+bt}\left(b\frac{x_2 - ct^2}{1+bt} + 2ct\right) \end{aligned}\right\} = 2c$$