MECHANICS OF SOLIDS

Jiří Plešek

Institute of Thermomechanics
Czech Academy of Sciences

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Lagrange description

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Four equations to solve: u_1, u_2, u_3, T.

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Kinematics

$$\mathbf{u} \quad \mapsto \quad \mathbf{z},\ \mathbf{F},\ J \quad \mapsto \quad \mathbf{U},\ \mathbf{E}$$

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input $\mathbf{E}(\mathbf{X},\tau)$, $T(\mathbf{X},\tau)$ and compute

- $\boldsymbol{\Sigma}(\mathbf{X},t) \quad \mapsto \quad \mathbf{P},\ \boldsymbol{\sigma}$
- $\mathbf{H}(\mathbf{X},t) \quad \mapsto \quad \mathbf{h}$
- $\psi(\mathbf{X},t) \quad \mapsto \quad \eta,\ u$

Remark: In general, functionals must be formed.

Thermoelasticity

Assumption

$$\exists \psi(\mathbf{E}, T): \quad \dot{\psi} = \frac{\partial \psi}{\partial E_{ij}} \dot{E}_{ij} + \frac{\partial \psi}{\partial T} \dot{T}$$

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Dissipation inequality

$$-\rho_0 \left(\eta + \frac{\partial \psi}{\partial T} \right) \dot{T} + \left(\Sigma_{ij} - \rho_0 \frac{\partial \psi}{\partial E_{ij}} \right) \dot{E}_{ij} - \frac{1}{T} \mathbf{H} \cdot \mathrm{Grad}\, T \geq 0$$

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$$\boxed{\eta = -\frac{\partial \psi}{\partial T} \quad \text{and} \quad \Sigma_{ij} = \rho_0 \frac{\partial \psi}{\partial E_{ij}}}$$

Caution: Conjugate stress $\mathbf{\Sigma}$ respective to $\mathbf{E}$ leaves the CE on output.

Fourier inequality

Reference configuration

$$\mathbf{H}\cdot\mathrm{Grad}\,T \leq 0$$

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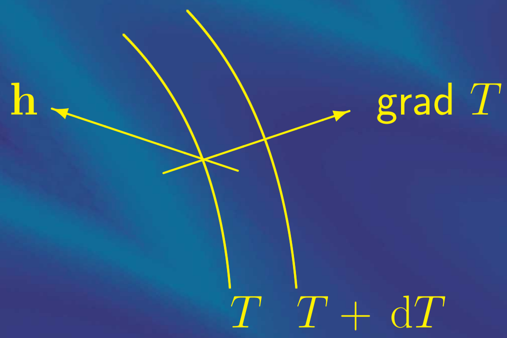
Current configuration

$$\mathbf{h}\cdot\mathrm{grad}\,T \leq 0$$

Fourier's 'law'



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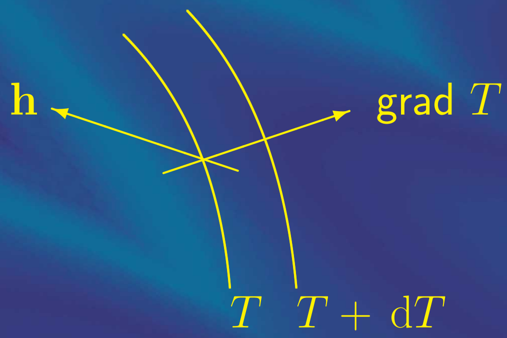

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$$\mathbf{h} = -\lambda \,\mathrm{grad}\, T$$

$$\lambda > 0 \text{ [W/mK]}$$

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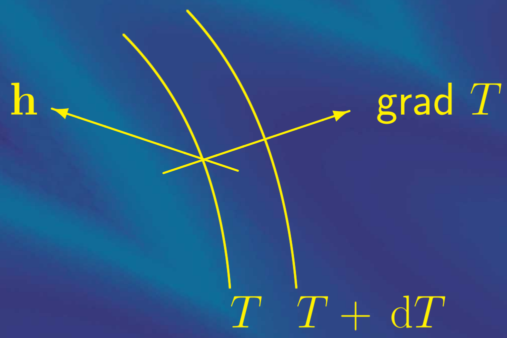

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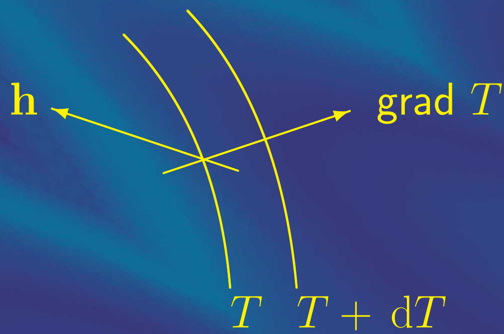

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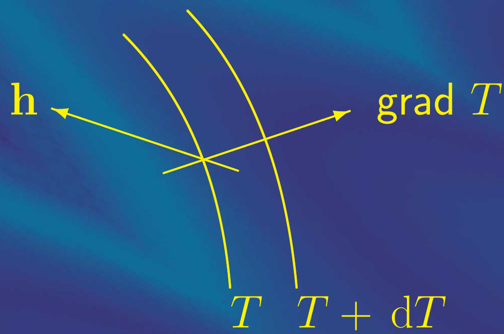

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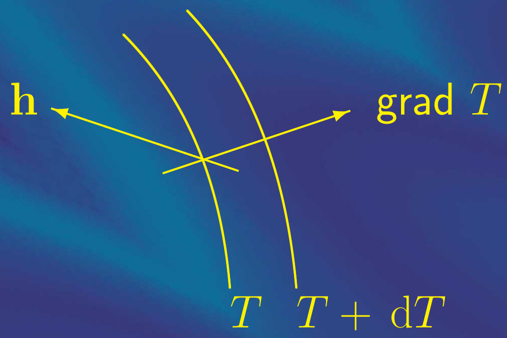

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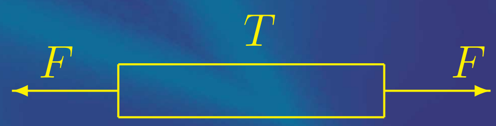
- In fact, Fourier's 'law' is merely a linear model.
- It violates the causality principle (parabolic vs. hyperbolic equation).
- Good for small gradients or steady state solutions.

Example 5: Uniaxial stress (1/4)



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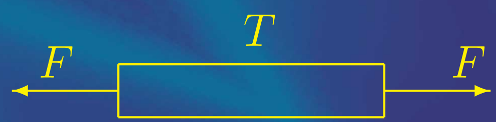

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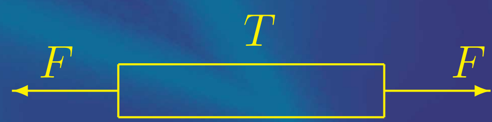

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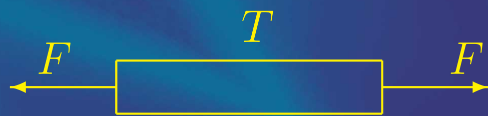

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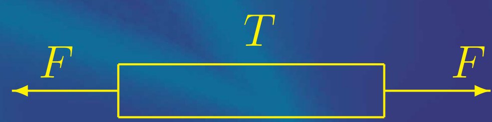

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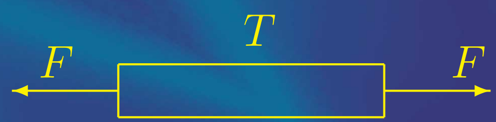

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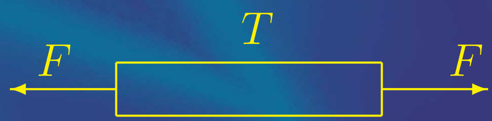

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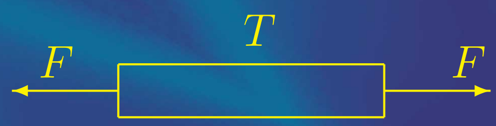

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$$\boxed{F = A\sigma_{11} = A_0 P_{11} = \lambda A_0 S_{11}}$$

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Remark: 2nd-PK does not permit direct computation of stress resultants.

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$$[e] = \frac{1}{2}(\,[F]^T[F] - [I]\,) = \frac{1}{2}\mathsf{diag}[\,(\lambda^2 - 1)\;\;(\lambda_r^2 - 1)\;\;(\lambda_r^2 - 1)\,]$$

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Uniaxial stress state

$$S_{11} = Ee_{11} = \frac{1}{2}E(\lambda^2 - 1)$$

Example 5: Uniaxial stress (2/4)

Constitutive assumption

$$\text{Hooke:}\quad \mathbf{e} \;\mapsto\; \mathbf{S}$$

GL-tensor

$$[e] = \frac{1}{2}(\,[F]^T[F] - [I]\,) = \frac{1}{2}\mathsf{diag}[\,(\lambda^2 - 1)\;\;(\lambda_r^2 - 1)\;\;(\lambda_r^2 - 1)\,]$$

Uniaxial stress state

$$S_{11} = Ee_{11} = \frac{1}{2}E(\lambda^2 - 1)$$

Stress resultant

$$\boxed{F = \frac{1}{2}EA_0\lambda(\lambda^2 - 1)}$$

Example 5: Uniaxial stress (3/4)

Small strain approximation, $\lambda \to 1$:

Taylor series

$$F(\lambda) = \underbrace{F(1)}_{0} + F'(1)(\lambda - 1) + \cdots$$

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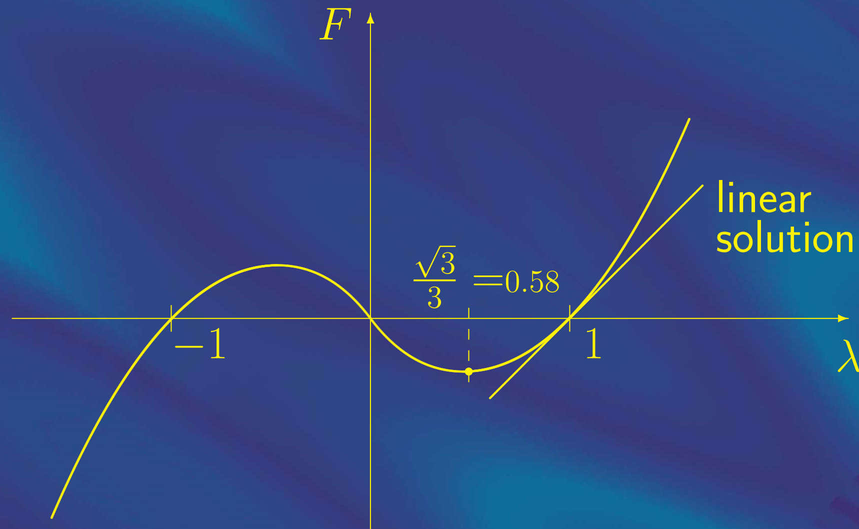
Hence

$$F \simeq EA_0(\lambda - 1) = EA_0\frac{l - l_0}{l_0} \quad \text{(checks)}$$

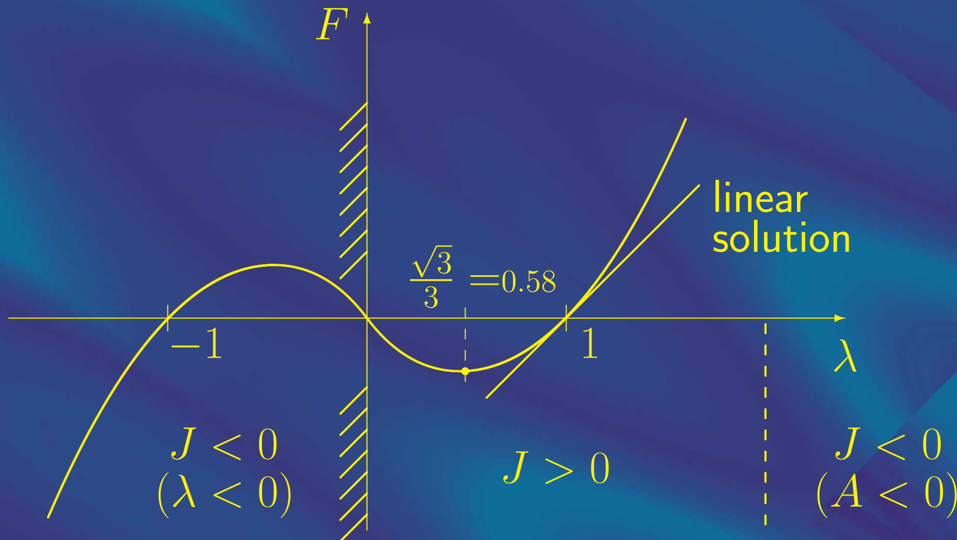
Example 5: Uniaxial stress (4/4)

Snap-through effect



Example 5: Uniaxial stress (4/4)

Snap-through effect



Example 5: Uniaxial stress (4/4)

Snap-through effect