Institute of Thermomechanics ASCR

TENSOR CHARACTER OF CONSTITUTIVE LAWS

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Contents

3D Elasticity

Free energy existence

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4th order tensor

$$\boxed{C_{ijkl} = \frac{\partial^2 \psi}{\partial \epsilon_{ij} \partial \epsilon_{kl}}} \qquad \text{(Hessian)}$$

Tangent stiffness symmetries

1. Minor symmetry

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(example: FE stiffness matrix)

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3. Material symmetry

isotropy group (E,ν)

Polar decomposition (1/2)

Let $[U]$ be 3×3 real symmetric matrix. Form the associated eigenproblem

$$[U]\{\varphi_k\} = \lambda_k\{\varphi_k\}\ , \quad \text{for } k = 1, 2, 3$$

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Orthonormal system

$$\{\varphi_i\}^T\{\varphi_j\} = \left\{ \begin{array}{l} 1, \ i = j \\ \\ 0, \ i \neq j \end{array} \right\} = \delta_{ij} \quad \text{(Kronecker)}$$

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Modal matrix: $[\Phi] = [\{\varphi_1\}\,\{\varphi_2\}\,\{\varphi_3\}]$, $\quad [\Phi]^T[\Phi] = [I]$ $\quad$ (orthonormal)

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$$\boxed{[U] = [\Phi][\Lambda][\Phi]^T}$$

Theorem: $[U]$ sym+def $\quad \Longleftrightarrow \quad \lambda_k > 0$

Note on quadratic forms

Example: energy of a linear system

$$P_2(x_1, x_2, \cdots, x_n) = U_{ij}x_ix_j = \{x\}^T[U]\{x\}$$

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$$U_{ij}x_ix_j = \cdots U_{12}x_1x_2 + U_{21}x_2x_1 \cdots = \cdots (U_{12} + U_{21})x_1x_2 \cdots$$

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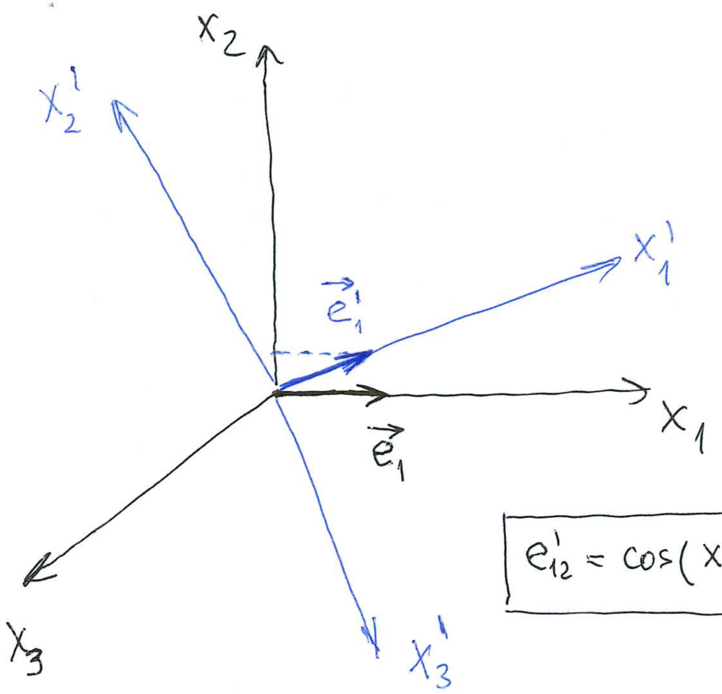
Convexity

$$\boxed{\forall\{x\} \neq \{0\} : \ \{x\}^T[U]\{x\} > 0}$$

We say $[U]$ is symmetric, positive definite (sym+def).

Cartesian system



$$\boxed{e_{12}' = \cos(x_1', x_2)}$$

column vector: $\{e_1'\} = \begin{Bmatrix} e_{11}' \\ e_{12}' \\ e_{13}' \end{Bmatrix}$

Orthogonal transformation

$[A] : x \longmapsto x'$, $\quad [A]^T[A] = [I]$

$$\boxed{\{x'\} = [A]\{x\}}$$

$$[A] = \begin{bmatrix} \underbrace{e_{11}' \quad e_{12}' \quad e_{13}'}_{\text{1st row} = \{e_1'\}^T} \\ \\ \\ \end{bmatrix}$$

Principal axes

Given a second order tensor

$$\tilde{\varepsilon} \longmapsto \varepsilon_{ij}\,,\ [\varepsilon]$$

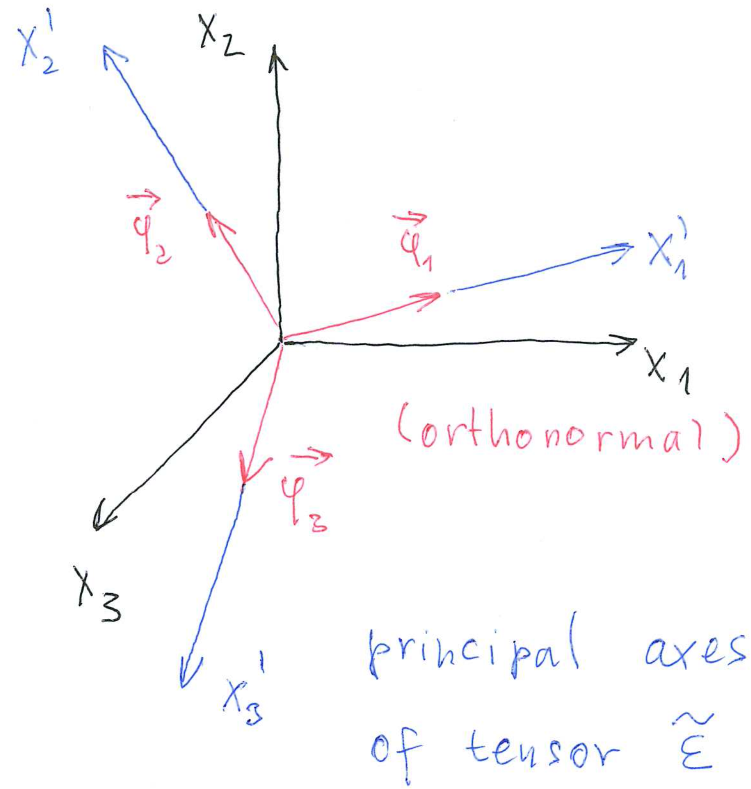
Eigenvalue problem for $k = 1, 2, 3$

$$[\varepsilon]\{\varphi_k\} = \lambda_k \{\varphi_k\}$$

principal values: $\varepsilon_k \equiv \lambda_k$

principal directions: $\{\varphi_k\}$

$$\vec{\varphi}_1 = \varphi_{11}\,\vec{e}_1 + \varphi_{12}\,\vec{e}_2 + \varphi_{13}\,\vec{e}_3$$



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Transformation matrix

$$[A] = \begin{bmatrix} \{\varphi_1\}^T \\ \{\varphi_2\}^T \\ \{\varphi_3\}^T \end{bmatrix} = [\Phi]^T$$

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$[\,\epsilon'\,] = [A][\epsilon][A]^T$

Principal axes

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2nd order tensor

$[\epsilon'] = [A][\epsilon][A]^T = [\Phi]^T[\epsilon][\Phi]$

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$$\boxed{[\,\epsilon'\,] = \mathrm{diag}\,[\,\epsilon_1\;\epsilon_2\;\epsilon_3\,]}$$

Invariants

Invariant function

$$I = f(\epsilon_{ij}) = f(\epsilon'_{ij}) = \text{scalar}$$

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Examples

- trace $\mathrm{tr}\,(\boldsymbol{\epsilon}) = \epsilon_{11} + \epsilon_{22} + \epsilon_{33} = \epsilon_{ii} = \epsilon'_{jj}$
- principal values $\epsilon_1,\ \epsilon_2,\ \epsilon_3$
- principal invariants $I_1,\ I_2,\ I_3$

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Power invariants

$$\boxed{\hat{I}_n = \frac{1}{n}\,\mathrm{tr}\,(\boldsymbol{\epsilon}^n)}$$

Isotropic material

Free energy

$$\psi(\epsilon_{ij}) = \psi(\epsilon_1, \epsilon_2, \epsilon_3)$$

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Chain rule

$$\sigma_{ij} = \frac{\partial \psi}{\partial \epsilon_{ij}}$$

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$$\sigma_{ij} = \frac{\partial \psi}{\partial \hat{I}_1}\frac{\partial \hat{I}_1}{\partial \epsilon_{ij}} + \frac{\partial \psi}{\partial \hat{I}_2}\frac{\partial \hat{I}_2}{\partial \epsilon_{ij}} + \frac{\partial \psi}{\partial \hat{I}_3}\frac{\partial \hat{I}_3}{\partial \epsilon_{ij}}$$

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Derivatives

$$\frac{\partial \hat{I}_1}{\partial \epsilon_{ij}} = \frac{\partial}{\partial \epsilon_{ij}}(\epsilon_{kk})$$

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General isotropic elasticity

Index notation

$$\boxed{\sigma_{ij} = \frac{\partial \psi}{\partial \hat{I}_1}\delta_{ij} + \frac{\partial \psi}{\partial \hat{I}_2}\epsilon_{ij} + \frac{\partial \psi}{\partial \hat{I}_3}\epsilon_{ik}\epsilon_{kj}}$$

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Direct notation

$$\boldsymbol{\sigma} = \frac{\partial \psi}{\partial \hat{I}_1}\mathbf{I} + \frac{\partial \psi}{\partial \hat{I}_2}\boldsymbol{\epsilon} + \frac{\partial \psi}{\partial \hat{I}_3}\boldsymbol{\epsilon}^2$$

Remark: Cayley-Hamilton theorem

Linear isotropic elasticity

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(Hooke's law)

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(Hooke's law)

$\lambda,\ \mu =$ Lamé

convexity: $E, K, G > 0,\ \nu \in (-1,\ 0.5)$