Institute of Thermomechanics ASCR

CONSERVATION LAWS I

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Contents

Lagrange description

$$\vec{u}(\vec{X}, t) \quad \text{in} \quad \Omega_0$$

$$\tilde{Z} = \text{Grad}\, \vec{u}$$

$$\vec{x} = \vec{X} + \vec{u} : \quad \Omega_0 \to \Omega_t$$

$$\vec{F} = \text{Grad}\, \vec{x} = \tilde{Z} + \tilde{I}$$

$$J = \det |\underline{F}| > 0$$

$$V_0 \to 0 : \quad J \simeq \frac{V_t}{V_0}$$

$$\tilde{F} = \tilde{R}\tilde{U} = \tilde{V}\tilde{R}$$

(based on polar decomposition)

$$\tilde{E} = f(\tilde{U}), \quad \tilde{A} = f(\tilde{V})$$

<u>example</u>: Green-Lagrange

$$\tilde{e} = \frac{1}{2}(\tilde{F}^T\tilde{F} - \tilde{I})$$

Euler description

$$\vec{v}(\vec{x},t) \quad \text{in} \quad \Omega \equiv \Omega_t$$

$$\tilde{L} = \operatorname{grad} \vec{v}$$

$$\tilde{L} = \tilde{D} + \tilde{W}$$

$$\dot{\tilde{T}} = \left(\frac{\partial \tilde{T}}{\partial t}\right)_X = \left(\frac{\partial \tilde{T}}{\partial t}\right)_x + (\operatorname{grad} \tilde{T})\,\vec{v}$$

$$\vec{a} = \dot{\vec{v}} = \left(\frac{\partial \vec{v}}{\partial t}\right)_x + \tilde{L}\,\vec{v}$$

$$\tilde{L} = \dot{\tilde{F}}\,\tilde{F}^{-1} \quad \Rightarrow \quad \dot{\tilde{e}} = \tilde{F}^T \tilde{D}\,\tilde{F}$$

$$\dot{\tilde{E}} = \mathcal{L}(\tilde{D})$$

<u>$t = 0$:</u> $\quad \tilde{D} = \dot{\tilde{\varepsilon}}$

1. Conservation of mass

Choose arbitrary volume

$$V_0 \subset \Omega_0 \;\rightarrow\; V_t \subset \Omega_t$$

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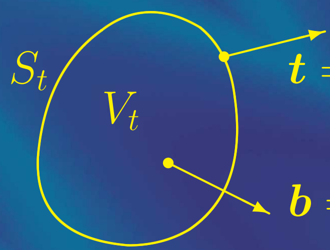
Choose arbitrary volume

$$V_0 \subset \Omega_0 \;\rightarrow\; V_t \subset \Omega_t$$

Same particles, same mass ...

$$\boxed{m(V_0) = m(V_t)}$$

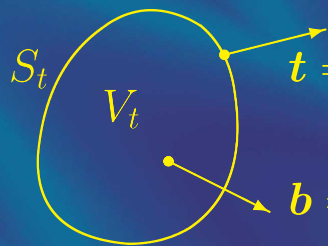
2. Conservation of momentum



$\boldsymbol{t} =$ stress vector in N/m^2

$\boldsymbol{b} =$ body force in N/m^3

2. Conservation of momentum



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Linear momentum definition

$$\mathbf{p}(V_t) = \int_{V_t} \rho \mathbf{v} \, \mathrm{d}V_t = \text{function}(t)$$

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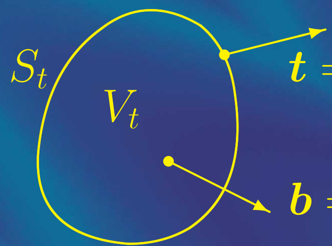

$\boldsymbol{t}$ = stress vector in $\mathrm{N/m^2}$

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Linear momentum definition

$$\mathbf{p}(V_t) = \int_{V_t} \rho \mathbf{v} \, \mathrm{d}V_t = \mathsf{function}(t)$$

$$\boxed{\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \int_{V_t} \mathbf{b} \, \mathrm{d}V_t + \int_{S_t} \mathbf{t} \, \mathrm{d}S_t}$$

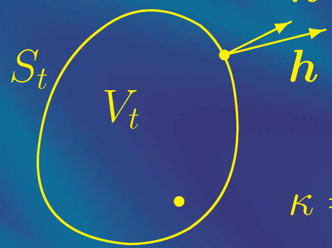
3. Conservation of angular momentum

Simple generalization

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_t} \rho \mathbf{x} \times \mathbf{v} \, \mathrm{d}V_t = \int_{V_t} \mathbf{x} \times \mathbf{b} \, \mathrm{d}V_t + \int_{S_t} \mathbf{x} \times \mathbf{t} \, \mathrm{d}S_t$$

- Caution: The cross product is carried out with respect to spatial coordinates at the current configuration.

4. Conservation of energy (1/2)



$\boldsymbol{n}$ = unit outward normal

$\boldsymbol{h}$ = heat flux in W/m^2

κ = heat source in W/m^3

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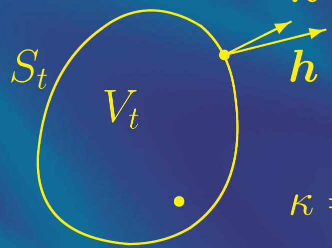


$\boldsymbol{n} =$ unit outward normal

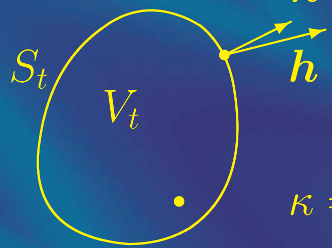
$\boldsymbol{h} =$ heat flux in $\mathrm{W/m^2}$

$\kappa =$ heat source in $\mathrm{W/m^3}$

Heat power input

$$\dot{Q}(V_t) = \int_{V_t} \kappa \, \mathrm{d}V_t - \int_{S_t} \mathbf{h} \cdot \mathbf{n} \, \mathrm{d}S_t$$

4. Conservation of energy (1/2)



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Mechanical power input

$$\dot{W}(V_t) = \int_{V_t} \mathbf{b} \cdot \mathbf{v} \, \mathrm{d}V_t + \int_{S_t} \mathbf{t} \cdot \mathbf{v} \, \mathrm{d}S_t$$

4. Conservation of energy (2/2)

Specific internal energy

$$\exists\, u : U(V_t) = \int_{V_t} \rho u \,\mathrm{d}V_t$$

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$$K(V_t) = \frac{1}{2}\int_{V_t} \rho \|\mathbf{v}\|^2 \,\mathrm{d}V_t$$

4. Conservation of energy (2/2)

Specific internal energy

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Kinetic energy

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Power conservation

$$\boxed{\dot{Q} + \dot{W} = \frac{\mathrm{d}}{\mathrm{d}t}(U + K)}$$

5. Clausius-Duhem inequality

Specific entropy

$$\exists\, \eta : S(V_t) = \int_{V_t} \rho\eta \,\mathrm{d}V_t$$

Heat form of CD inequality

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t} S \geq \int_{V_t} \frac{\kappa}{T} \,\mathrm{d}V_t - \int_{S_t} \frac{\mathbf{h}\cdot\mathbf{n}}{T} \,\mathrm{d}S_t}$$

Remark: extensive vs. intensive quantities.

1. Conservation of mass

Jacobian interpretation

Density

$$m(V_0) = \int_{V_0} \rho_0 \, \mathrm{d}V_0$$

Jacobian interpretation

Densities

$$m(V_0) = \int_{V_0} \rho_0 \, \mathrm{d}V_0 \,, \qquad m(V_t) = \int_{V_t} \rho \, \mathrm{d}V_t$$

Jacobian interpretation

Densities

$$m(V_0) = \int_{V_0} \rho_0 \, \mathrm{d}V_0 \, , \qquad m(V_t) = \int_{V_t} \rho \, \mathrm{d}V_t = \int_{V_0} J \rho \, \mathrm{d}V_0$$

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By the mass conservation law

$$\int_{V_0} (\rho_0 - J\rho) \,\mathrm{d}V_0 = 0$$

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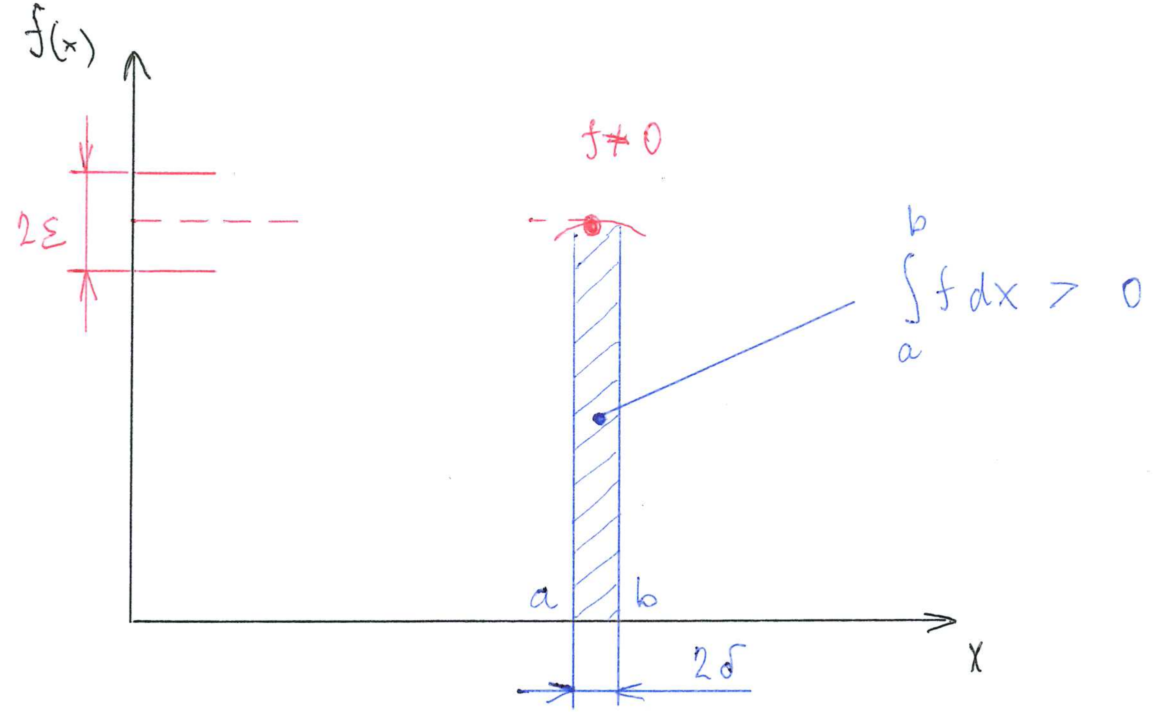
Local form

$$\boxed{J\rho = \rho_0}$$

Continuous functions:



Cauchy:

$$\forall \varepsilon > 0,\ \exists \delta > 0 :\quad \forall x \in U(x_0, \delta) \Rightarrow f(x) \in U(f(x_0), \varepsilon)$$

General guidlines

- Write a conservation law in the integral (total) form.
- Integral denomination: all the integrals over the same domain.
- Remove integral signs to obtain differential (local) form.
- Check on whether the law indeed holds for an arbitrary subdomain.

FEM

Mass matrix in a reference configuration

$$[M] = \int_{V_0} \rho_0 [N]^T [N] \, \mathrm{d}V_0$$

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- The mass matrix remains invariant under deformation, small or large.

Reynolds' transport theorem

Given a general field $\mathbf{T}$ = scalar, vector, tensor, ...

$$\mathbf{T} = \phi(\mathbf{X}, t) = \varphi(\mathbf{x}, t)$$

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Theorem

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_t} \rho \mathbf{T} \,\mathrm{d}V_t = \int_{V_t} \rho \dot{\mathbf{T}} \,\mathrm{d}V_t}$$

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Continuity equation (1/2)

The initial density is not a function of time . . .

$$\dot{\rho}_0 = (J\rho)^{\cdot} = \dot{J}\rho + J\dot{\rho} = 0$$

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Theorem (proved later)

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Material derivative variant

$$\boxed{\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0}$$

Continuity equation (2/2)

Using local derivative

$$\left(\frac{\partial \rho}{\partial t}\right)_x + (\operatorname{grad} \rho)\mathbf{v} + \rho \operatorname{div} \mathbf{v} = 0$$

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$$\frac{\partial \rho}{\partial x_j} v_j + \rho \frac{\partial v_j}{x_j} = \frac{\partial}{\partial x_j}(\rho v_j)$$

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Compare to

$$\left(\frac{\partial \rho}{\partial t}\right)_X + \rho \operatorname{div}\mathbf{v} = 0$$

Cofactor (1/2)

Definition for a 3×3 matrix

$$\bar{F}_{ij} = (-1)^{i+j}\text{subdet}|F_{ij}|$$

Cofactor (1/2)

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$$\bar{F}_{ij} = (-1)^{i+j}\mathsf{subdet}|F_{ij}|$$

Example

$$\mathsf{subdet}|F_{21}| = \begin{vmatrix} \circ & F_{12} & F_{13} \\ \bullet & \circ & \circ \\ \circ & F_{32} & F_{33} \end{vmatrix}$$

Cofactor (1/2)

Definition for a 3×3 matrix

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Examples

$$\bar{F}_{21} = -\det\begin{vmatrix} F_{12} & F_{13} \\ F_{32} & F_{33} \end{vmatrix}, \quad \bar{F}_{22} = \det\begin{vmatrix} F_{11} & F_{13} \\ F_{31} & F_{33} \end{vmatrix}, \quad \bar{F}_{23} = -\det\begin{vmatrix} F_{11} & F_{12} \\ F_{31} & F_{32} \end{vmatrix}$$

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Row expansion

$$J = \det |F| = F_{21}\bar{F}_{21} + F_{22}\bar{F}_{22} + F_{23}\bar{F}_{23}$$

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Row expansion

$$J = \det|F| = F_{21}\bar{F}_{21} + F_{22}\bar{F}_{22} + F_{23}\bar{F}_{23}$$

Important: no cofactor depends on either of F_{21}, F_{22}, F_{23}.

Cofactor (2/2)

Differentiating

$$\frac{\partial J}{\partial F_{21}} = \bar{F}_{21} \quad \Rightarrow \quad \frac{\partial J}{\partial F_{ij}} = \bar{F}_{ij}$$

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$$F_{ij}^{-1} = \frac{1}{J}\bar{F}_{ji}$$

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Cramer's rule

$$F^{-1}_{ij} = \frac{1}{J}\bar{F}_{ji}$$

Hence

$$\boxed{\frac{\partial J}{\partial F_{ij}} = J F^{-1}_{ji}}$$

Cofactor (2/2)

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Hence

$$\boxed{\frac{\partial J}{\partial F_{ij}} = J F^{-1}_{ji}}$$

Finally

$$\dot{J} = \frac{\partial J}{\partial F_{ij}}\dot{F}_{ij}$$

Cofactor (2/2)

Differentiating

$$\frac{\partial J}{\partial F_{21}} = \bar{F}_{21} \quad \Rightarrow \quad \frac{\partial J}{\partial F_{ij}} = \bar{F}_{ij}$$

Cramer's rule

$$F^{-1}_{ij} = \frac{1}{J}\bar{F}_{ji}$$

Hence

$$\boxed{\frac{\partial J}{\partial F_{ij}} = J F^{-1}_{ji}}$$

Finally

$$\dot{J} = \frac{\partial J}{\partial F_{ij}}\dot{F}_{ij} = (J F^{-1}_{ji})\dot{F}_{ij}$$

Cofactor (2/2)

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$$\boxed{\frac{\partial J}{\partial F_{ij}} = J F^{-1}_{ji}}$$

Finally

$$\dot{J} = \frac{\partial J}{\partial F_{ij}}\dot{F}_{ij} = (J F^{-1}_{ji})\dot{F}_{ij} = J\dot{F}_{ij}F^{-1}_{ji}$$

Cofactor (2/2)

Differentiating

$$\frac{\partial J}{\partial F_{21}} = \bar{F}_{21} \quad \Rightarrow \quad \frac{\partial J}{\partial F_{ij}} = \bar{F}_{ij}$$

Cramer's rule

$$F^{-1}_{ij} = \frac{1}{J}\bar{F}_{ji}$$

Hence

$$\boxed{\frac{\partial J}{\partial F_{ij}} = J F^{-1}_{ji}}$$

Finally

$$\dot{J} = \frac{\partial J}{\partial F_{ij}}\dot{F}_{ij} = (J F^{-1}_{ji})\dot{F}_{ij} = J\dot{F}_{ij}F^{-1}_{ji} = J L_{ii}$$

Cofactor (2/2)

Differentiating

$$\frac{\partial J}{\partial F_{21}} = \bar{F}_{21} \quad \Rightarrow \quad \frac{\partial J}{\partial F_{ij}} = \bar{F}_{ij}$$

Cramer's rule

$$F^{-1}_{ij} = \frac{1}{J}\bar{F}_{ji}$$

Hence

$$\boxed{\frac{\partial J}{\partial F_{ij}} = J F^{-1}_{ji}}$$

Finally

$$\dot{J} = \frac{\partial J}{\partial F_{ij}}\dot{F}_{ij} = (J F^{-1}_{ji})\dot{F}_{ij} = J \dot{F}_{ij} F^{-1}_{ji} = J L_{ii} = J \frac{\partial v_i}{\partial x_i}$$

Cofactor (2/2)

Differentiating

$$\frac{\partial J}{\partial F_{21}} = \bar{F}_{21} \quad \Rightarrow \quad \frac{\partial J}{\partial F_{ij}} = \bar{F}_{ij}$$

Cramer's rule

$$F^{-1}_{ij} = \frac{1}{J}\bar{F}_{ji}$$

Hence

$$\boxed{\frac{\partial J}{\partial F_{ij}} = J F^{-1}_{ji}}$$

Finally

$$\dot{J} = \frac{\partial J}{\partial F_{ij}}\dot{F}_{ij} = (J F^{-1}_{ji})\dot{F}_{ij} = J \dot{F}_{ij} F^{-1}_{ji} = J L_{ii} = J \frac{\partial v_i}{\partial x_i} = J \operatorname{div} \mathbf{v}$$