Institute of Thermomechanics ASCR

CONSERVATION LAWS II

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- Conservation of energy

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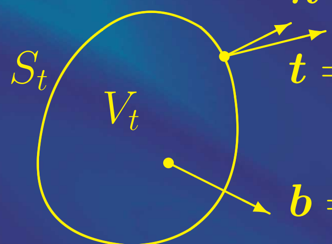
Spatial description

In the current configuration . . .

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \int_{V_t} \mathbf{b}\,\mathrm{d}V_t + \int_{S_t} \mathbf{t}\,\mathrm{d}S_t$$



$\boldsymbol{n}$ = unit outward normal

$\boldsymbol{t}$ = stress vector in $\mathrm{N/m^2}$

$\boldsymbol{b}$ = body force in $\mathrm{N/m^3}$

Spatial description

In the current configuration . . .

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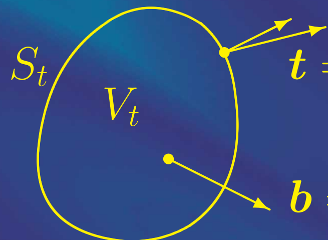

$\boldsymbol{n}$ = unit outward normal

$\boldsymbol{t}$ = stress vector in N/m^2

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By the Reynolds theorem . . .

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_t} \rho\mathbf{v}\,\mathrm{d}V_t = \int_{V_t} \rho\dot{\mathbf{v}}\,\mathrm{d}V_t = \int_{V_t} \rho\mathbf{a}\,\mathrm{d}V_t$$

Cauchy stress

The Cauchy theorem

$$\boxed{\exists \boldsymbol{\sigma} \text{ in } \Omega_t : \mathbf{t} = \boldsymbol{\sigma}\mathbf{n}}$$

- The stress tensor is defined in Ω_t independent of the choice of $S_t(V_t)$.

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Definition of the spatial divergence operation

$$\operatorname{div} \boldsymbol{\sigma} = \frac{\partial \sigma_{ij}}{\partial x_j} \mathbf{e}_i$$

Spatial equations of motion

Putting everything together ...

$$\int_{V_t} \rho \mathbf{a} \, \mathrm{d}V_t = \int_{V_t} \mathbf{b} \, \mathrm{d}V_t + \int_{V_t} \operatorname{div} \boldsymbol{\sigma} \, \mathrm{d}V_t$$

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It holds for every V_t, therefore

$$\boxed{\operatorname{div} \boldsymbol{\sigma} + \mathbf{b} = \rho \mathbf{a}} \qquad \text{in} \quad \Omega_t \equiv \Omega$$

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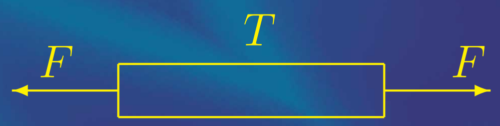
From the conservation of angular momentum it follows that $\boldsymbol{\sigma}$ is symmetric.

Example 5: Uniaxial stress



A = current crossection

l = current length

$$\sigma_{11} = \frac{F}{A}$$ true stress (Cauchy)

$$P_{11} = \frac{F}{A_0}$$ nominal stress (1st Piola-Kirchhoff)

Example 5: Uniaxial stress

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$$\sigma_{11} = \frac{F}{A}$$ true stress (Cauchy)

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Idea: for $\Delta S_0 \subset S_0 \;\rightarrow\; \Delta S_t \subset S_t$ set the nominal stress vector $\mathbf{T}$ such that

$$\int_{\Delta S_t} \mathbf{t}\, \mathrm{d}S_t = \int_{\Delta S_0} \mathbf{T}\, \mathrm{d}S_0$$

Transformation of inertial terms

Momentum

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \int_{V_t} \rho \mathbf{a} \, \mathrm{d}V_t = \int_{V_0} J \rho \mathbf{a} \, \mathrm{d}V_0$$

Transformation of inertial terms

Momentum

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \int_{V_t} \rho\mathbf{a}\,\mathrm{d}V_t = \int_{V_0} \underbrace{J\rho}_{\rho_0}\,\mathbf{a}\,\mathrm{d}V_0$$

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Body force

$$\int_{V_t} \mathbf{b} \,\mathrm{d}V_t = \int_{V_0} \underbrace{J\mathbf{b}}_{\mathbf{B}} \,\mathrm{d}V_0 = \int_{V_0} \mathbf{B} \,\mathrm{d}V_0$$

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Body force in the reference configuration

$$\boxed{\mathbf{B} = J\mathbf{b}}$$

Material equations of motion

$$\int_{V_0} \rho_0 \mathbf{a} \, \mathrm{d}V_0 = \int_{V_0} \mathbf{B} \, \mathrm{d}V_0 + \int_{S_0} \mathbf{T} \, \mathrm{d}S_0$$

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By the Cauchy theorem

$$\exists \mathbf{P} \text{ in } \Omega_0 : \mathbf{T} = \mathbf{P}\mathbf{N}$$

where $\mathbf{N}$ is the unit outward normal respective to S_0.

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where $\mathbf{N}$ is the unit outward normal respective to S_0.

Therefore

$$\boxed{\mathrm{Div}\, \mathbf{P} + \mathbf{B} = \rho_0 \, \mathbf{a}} \qquad \text{in} \quad \Omega_0$$

Nanson's formula

Given a general field $\mathbf{T}$ = scalar, vector, tensor, ...

$$\mathbf{T} = \phi(\mathbf{X}, t) = \varphi(\mathbf{x}, t)$$

It holds

$$\boxed{\int_{\Delta S_t} \mathbf{T}\mathbf{n}\,\mathrm{d}S_t = \int_{\Delta S_0} J\mathbf{T}\mathbf{F}^{-T}\mathbf{N}\,\mathrm{d}S_0}$$

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Tentative proof for a closed surface

$$\int_{S_t} \varphi n_j \,\mathrm{d}S_t =$$

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$$= \int_{V_0} \underline{\frac{\partial}{\partial X_k}(J\phi F^{-1}_{kj})}\,\mathrm{d}V_0$$

Gabrio Piola (1794-1850): Piola identity (underlined), Piola transformation

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1st Piola-Kirchhoff stress tensor

Component notation

$$\int_{\Delta S_t} t_i \, \mathrm{d}S_t =$$

1st Piola-Kirchhoff stress tensor

Component notation

$$\int_{\Delta S_t} t_i \,\mathrm{d}S_t = \int_{\Delta S_t} \sigma_{ij} n_j \,\mathrm{d}S_t$$

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Definition

$$P_{ik} = J\sigma_{ij}F^{-1}_{kj} \quad \Rightarrow \quad \boxed{\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-T}}$$

nonsymmetric

1st Piola-Kirchhoff stress tensor

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Definitions

$$P_{ik} = J\sigma_{ij}F^{-1}_{kj} \quad \Rightarrow \quad \boxed{\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-T}}$$

nonsymmetric

$$T_i = P_{ik}N_k \quad \Rightarrow \quad \mathbf{T} = \mathbf{PN}$$

1st Piola-Kirchhoff stress tensor

Component notation

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Definitions

$$P_{ik} = J\sigma_{ij}F^{-1}_{kj} \quad \Rightarrow \quad \boxed{\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-T}}$$ nonsymmetric

$$T_i = P_{ik}N_k \quad \Rightarrow \quad \mathbf{T} = \mathbf{P}\mathbf{N}$$

- Note that the Cauchy relation was derived by transformation.

4. Conservation of energy

$$\dot{Q} + \dot{W} = \frac{\mathrm{d}}{\mathrm{d}t}(U + K)$$

Kinetic energy

Owing to Reynold's theorem ...

$$\frac{\mathrm{d}}{\mathrm{d}t} K = \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \int_{V_t} \rho \|\mathbf{v}\|^2 \, \mathrm{d}V_t$$

Kinetic energy

Owing to Reynold's theorem ...

$$\frac{\mathrm{d}}{\mathrm{d}t}K = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{V_t}\rho\|\mathbf{v}\|^2\,\mathrm{d}V_t = \frac{1}{2}\int_{V_t}\rho(\mathbf{v}\cdot\mathbf{v})^{\cdot}\,\mathrm{d}V_t$$

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$$= \int_{V_t}\rho\dot{\mathbf{v}}\cdot\mathbf{v}\,\mathrm{d}V_t$$

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$$= \int_{V_t}\rho\dot{\mathbf{v}}\cdot\mathbf{v}\,\mathrm{d}V_t = \int_{V_t}\rho\mathbf{a}\mathbf{v}\,\mathrm{d}V_t$$

Thus, we have

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}K = \int_{V_t}\rho\mathbf{a}\mathbf{v}\,\mathrm{d}V_t}$$

Mechanical power (1/2)

Definition

$$\dot{W} = \int_{V_t} \mathbf{b} \cdot \mathbf{v} \, \mathrm{d}V_t + \int_{S_t} \mathbf{t} \cdot \mathbf{v} \, \mathrm{d}S_t$$

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Surface integral

$$\int_{S_t} \mathbf{t} \cdot \mathbf{v} \, \mathrm{d}S_t = \int_{S_t} t_i v_i \, \mathrm{d}S_t$$

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$$\int_{S_t} \mathbf{t} \cdot \mathbf{v} \, \mathrm{d}S_t = \int_{S_t} t_i v_i \, \mathrm{d}S_t = \int_{S_t} \sigma_{ij} n_j v_i \, \mathrm{d}S_t$$

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$$= \int_{V_t} \frac{\partial \sigma_{ij}}{\partial x_j} v_i \, \mathrm{d}V_t + \int_{V_t} \sigma_{ij} \frac{\partial v_i}{\partial x_j} \, \mathrm{d}V_t$$

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$$= \int_{V_t} \frac{\partial \sigma_{ij}}{\partial x_j} v_i \, \mathrm{d}V_t + \int_{V_t} \sigma_{ij} \frac{\partial v_i}{\partial x_j} \, \mathrm{d}V_t = \int_{V_t} \mathbf{v} \cdot \operatorname{div} \sigma \, \mathrm{d}V_t + \int_{V_t} \sigma_{ij} L_{ij} \, \mathrm{d}V_t$$

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Surface integral

$$\int_{S_t} \mathbf{t} \cdot \mathbf{v} \, \mathrm{d}S_t = \int_{S_t} t_i v_i \, \mathrm{d}S_t = \int_{S_t} \sigma_{ij} n_j v_i \, \mathrm{d}S_t = \int_{V_t} \frac{\partial}{\partial x_j}(\sigma_{ij} v_i) \, \mathrm{d}V_t$$

$$= \int_{V_t} \frac{\partial \sigma_{ij}}{\partial x_j} v_i \, \mathrm{d}V_t + \int_{V_t} \sigma_{ij} \frac{\partial v_i}{\partial x_j} \, \mathrm{d}V_t = \int_{V_t} \mathbf{v} \cdot \mathrm{div}\, \sigma \, \mathrm{d}V_t + \int_{V_t} \sigma_{ij} L_{ij} \, \mathrm{d}V_t$$

Inserting . . .

$$\dot{W} = \int_{V_t} (\mathbf{b} + \mathrm{div}\, \sigma) \cdot \mathbf{v} \, \mathrm{d}V_t + \int_{V_t} \sigma_{ij} L_{ij} \, \mathrm{d}V_t$$

Mechanical power (1/2)

Definition

$$\dot{W} = \int_{V_t} \mathbf{b} \cdot \mathbf{v} \, \mathrm{d}V_t + \int_{S_t} \mathbf{t} \cdot \mathbf{v} \, \mathrm{d}S_t$$

Surface integral

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Inserting . . .

$$\dot{W} = \int_{V_t} \underbrace{(\mathbf{b} + \mathrm{div}\, \sigma)}_{\rho \mathbf{a}} \cdot \mathbf{v} \, \mathrm{d}V_t + \int_{V_t} \sigma_{ij} L_{ij} \, \mathrm{d}V_t$$

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Inserting ...

$$\dot{W} = \int_{V_t} \underbrace{(\mathbf{b} + \mathrm{div}\,\sigma)}_{\rho\mathbf{a}}\cdot\mathbf{v}\,\mathrm{d}V_t + \int_{V_t} \sigma_{ij} L_{ij}\,\mathrm{d}V_t = \frac{\mathrm{d}}{\mathrm{d}t} K + \int_{V_t} \sigma_{ij} L_{ij}\,\mathrm{d}V_t$$

Mechanical power (2/2)

Additive decomposition

$$\sigma_{ij}L_{ij} = \sigma_{ij}(D_{ij} + W_{ij}) = \sigma_{ij}D_{ij}$$

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At $t = 0:\quad \boldsymbol{\sigma}:\mathbf{D} = \boldsymbol{\sigma}:\dot{\boldsymbol{\epsilon}} = \sigma_{ij}\dot{\epsilon}_{ij}$ (small strain approximation)

Spatial equation of heat conduction

Thus, even for fast loadings . . .

$$\boxed{\dot{Q} + \int_{V_t} \boldsymbol{\sigma} : \mathbf{D} \, \mathrm{d}V_t = \frac{\mathrm{d}}{\mathrm{d}t} U}$$

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Heat conduction with a thermomechanical coupling term

$$\boxed{\kappa - \operatorname{div} \mathbf{h} + \boldsymbol{\sigma} : \mathbf{D} = \rho \dot{u}} \qquad \text{in} \quad \Omega_t \equiv \Omega$$