Institute of Thermomechanics ASCR

LAGRANGE DESCRIPTION I

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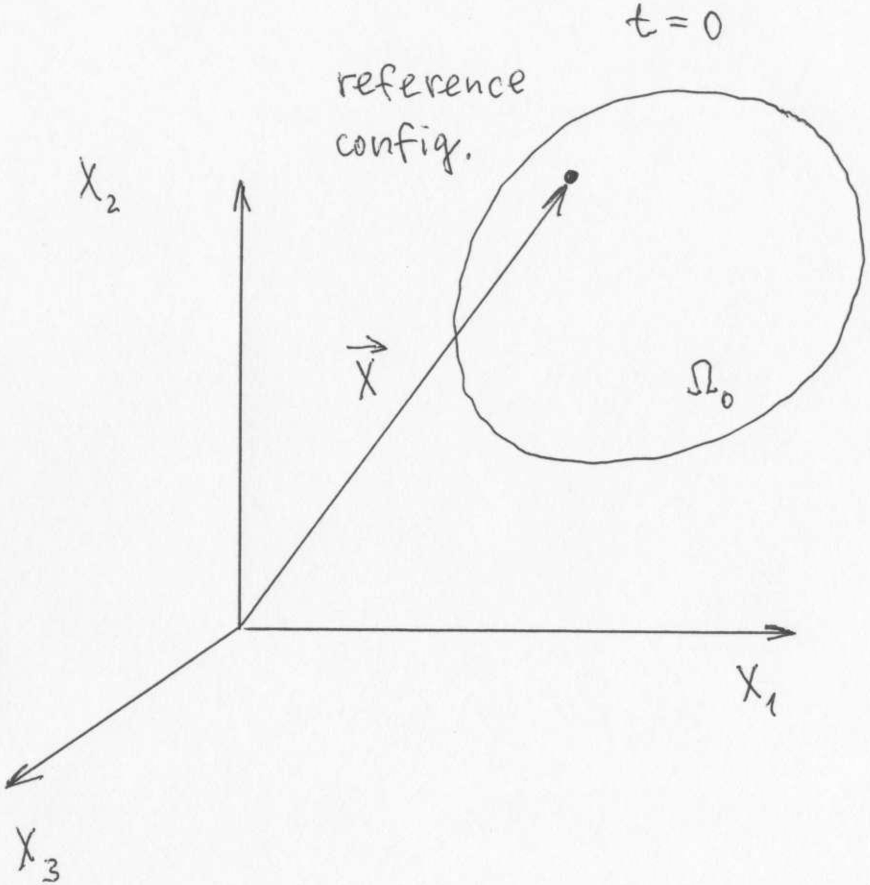
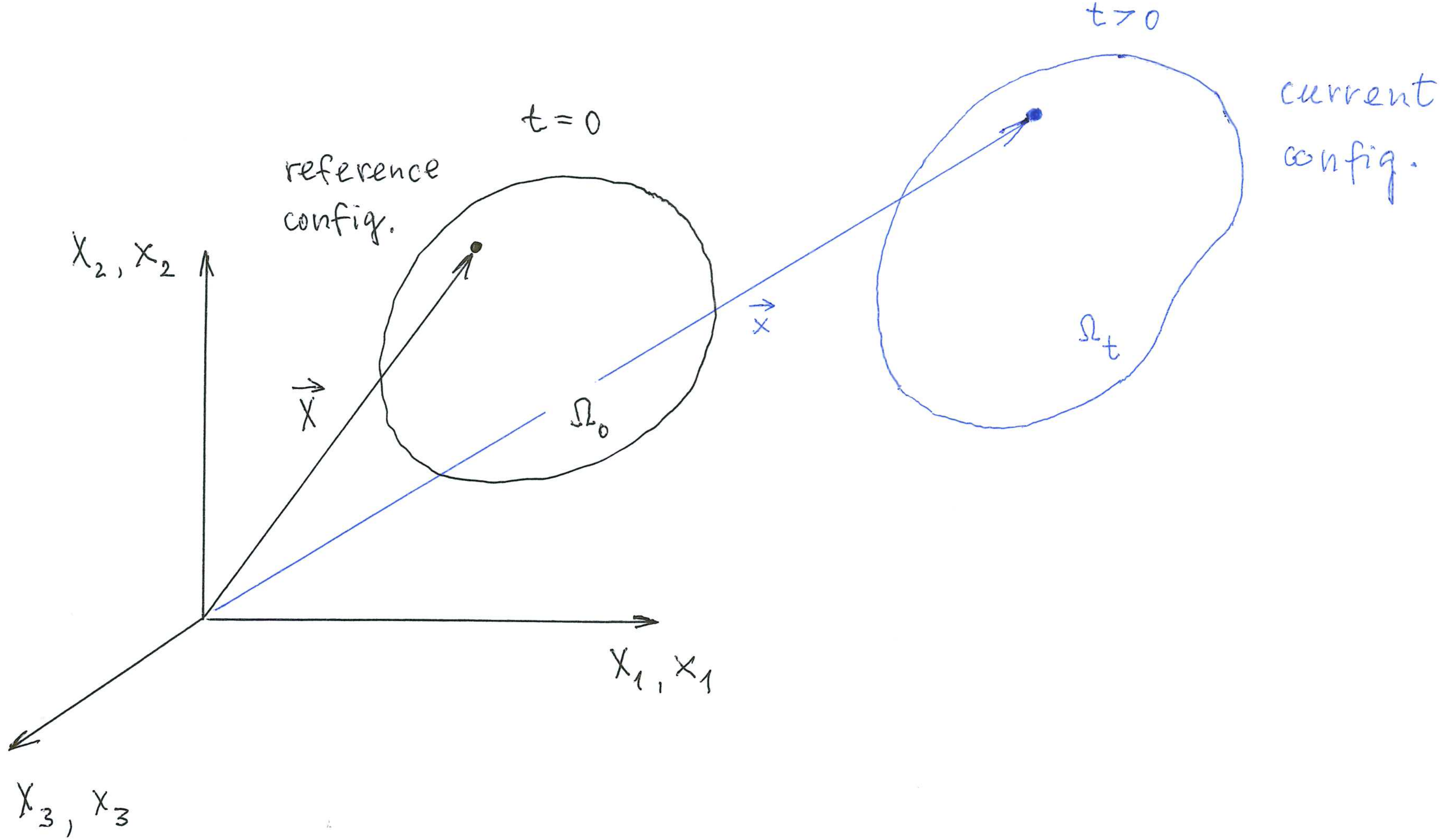
Contents

- Kinematic layout
- Displacement gradient
- Deformation gradient
- Jacobian
- Green-Lagrange strain tensor

KINEMATICS

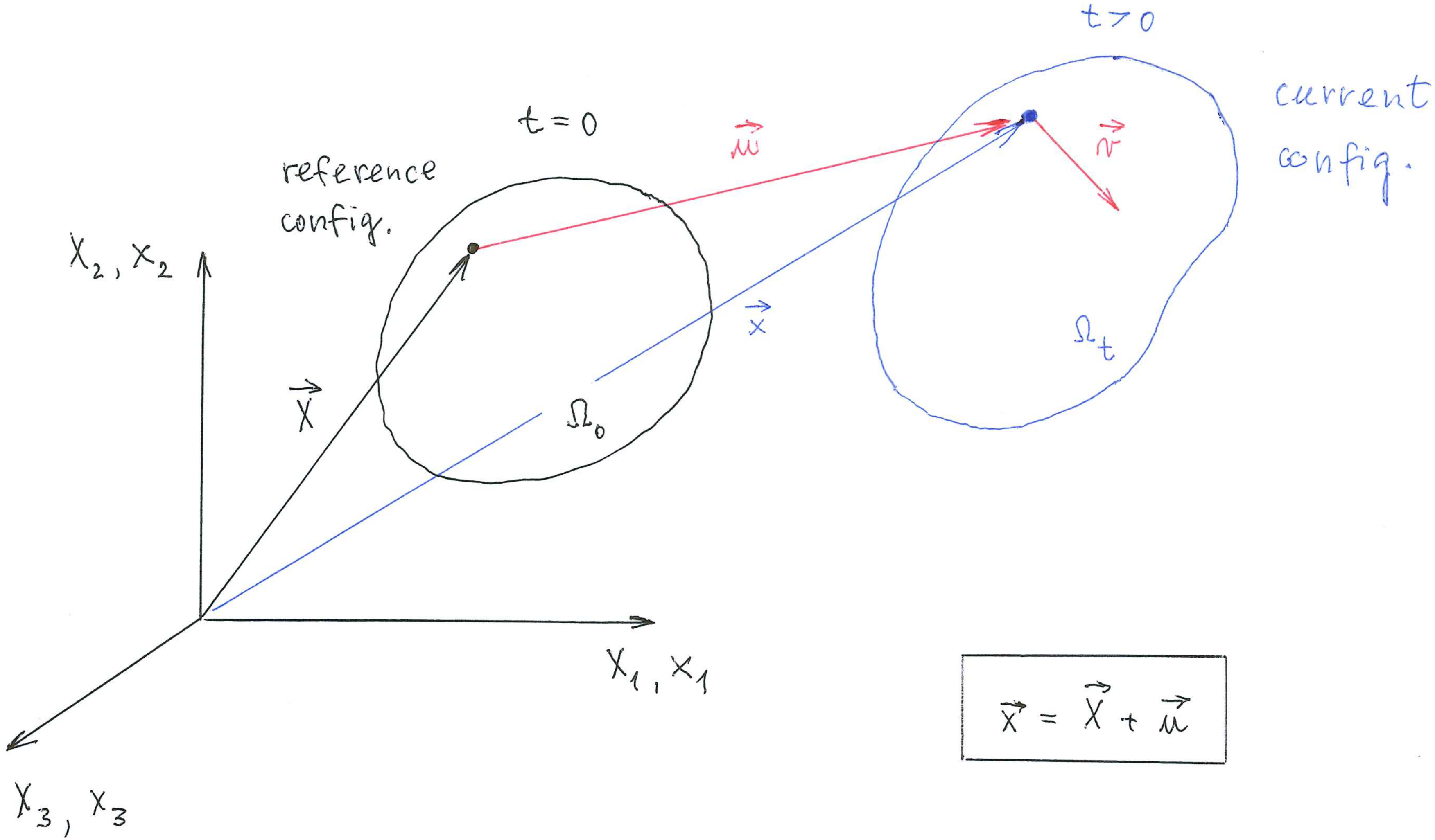


KINEMATICS



$t=0$
reference config.
$t>0$
current config.
X_2, x_2
$\vec{X}$
$\vec{x}$
Ω_0
Ω_t
X_1, x_1
X_3, x_3

KINEMATICS



$$\boxed{\vec{x} = \vec{X} + \vec{u}}$$

Nomenclature

$\mathbf{X} \in \Omega_0$	material point (particle)
$\mathbf{x} \in \mathrm{R}^3$	spatial point

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$\mathbf{X} \in \Omega_0$	material point (particle)
$\mathbf{x} \in \mathrm{R}^3$	spatial point

X_i	material coordinates (Lagrange)
x_j	spatial coordinates (Euler)

$\mathbf{u}(\mathbf{X}, t)$	material (Lagrangian) field in Ω_0
$\mathbf{v}(\mathbf{x}, t)$	spatial (Eulerian) field in $\Omega \subset \mathrm{R}^3$

Lagrangian fields

Primary field: $\mathbf{u}(\mathbf{X}, t) \;=\;$ displacement field in Ω_0

$$\boxed{\mathbf{x}(\mathbf{X}, t) = \mathbf{X} + \mathbf{u}(\mathbf{X}, t)}$$

Lagrangian fields

Primary field: $\mathbf{u}(\mathbf{X}, t) \; = \;$ displacement field in Ω_0

$$\boxed{\mathbf{x}(\mathbf{X}, t) = \mathbf{X} + \mathbf{u}(\mathbf{X}, t)}$$

Domain mapping

$$\mathbf{x}(\mathbf{X}, t) : \Omega_0 \to \Omega_t \quad \text{(regular)}$$

Inverse mapping

$$\exists\, \mathbf{X}(\mathbf{x}, t) : \Omega_t \to \Omega_0$$

Initial condition

$$\mathbf{x}(\mathbf{X}, 0) = \mathbf{X}$$

Displacement gradient

Component definition

$$z_{ij} = \frac{\partial u_i}{\partial X_j} \quad \text{(transforms as a second order tensor)}$$

Matrix notation

$$[z] = \begin{bmatrix} \dfrac{\partial u_1}{\partial X_1} & \dfrac{\partial u_1}{\partial X_2} & \dfrac{\partial u_1}{\partial X_3} \\ \dfrac{\partial u_2}{\partial X_1} & \dfrac{\partial u_2}{\partial X_2} & \dfrac{\partial u_2}{\partial X_3} \\ \dfrac{\partial u_3}{\partial X_1} & \dfrac{\partial u_3}{\partial X_2} & \dfrac{\partial u_3}{\partial X_3} \end{bmatrix}$$

Displacement gradient

Component definition

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Direct (covariant) notation

$$\boxed{\mathbf{z} = \operatorname{Grad} \mathbf{u}}$$

Deformation gradient

Component definition

$$F_{ij} = \frac{\partial x_i}{\partial X_j} \quad \text{(transforms as a second order tensor)}$$

Direct notation

$$\boxed{\mathbf{F} = \operatorname{Grad} \mathbf{x}}$$

Deformation gradient

Component definition

$$F_{ij} = \frac{\partial x_i}{\partial X_j} \quad \text{(transforms as a second order tensor)}$$

Direct notation

$$\boxed{\mathbf{F} = \operatorname{Grad} \mathbf{x}}$$

Relation between gradients

$$F_{ij} = \frac{\partial x_i}{\partial X_j} = \frac{\partial}{\partial X_j}(X_i + u_i) = \delta_{ij} + z_{ij}$$

$$\boxed{\mathbf{F} = \mathbf{z} + \mathbf{I}}$$

Jacobian

Jacobi matrix

$$J_{ij} = \frac{\partial x_i}{\partial X_j} = F_{ij}$$

Jacobian

$$J = \det |F| \neq 0 \quad \text{everywhere in } \Omega_0 \text{ (regular)}$$

Jacobian

Jacobi matrix

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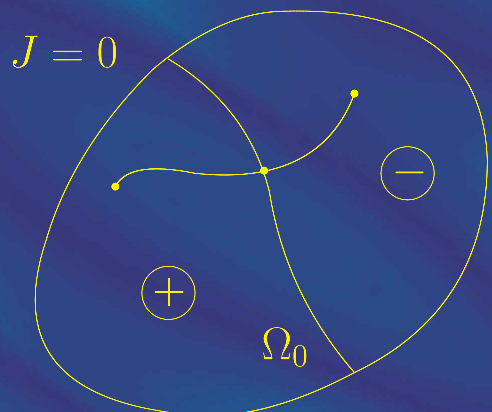

$J > 0$ or $J < 0$ everywhere in Ω_0

Volume

Choose arbitrarily

$$V_0 \subset \Omega_0 \;\rightarrow\; V_t \subset \Omega_t$$

Volume

Choose arbitrarily

$$V_0 \subset \Omega_0 \ \rightarrow \ V_t \subset \Omega_t$$

Volumetric integral

$$V_t = \int_{V_t} \mathrm{d}V_t = \int_{V_0} J \, \mathrm{d}V_0 > 0$$

Hence

$$\boxed{J = \det |F| > 0} \quad \text{in } \Omega_0$$

Volume

Choose arbitrarily

$$V_0 \subset \Omega_0 \;\rightarrow\; V_t \subset \Omega_t$$

Volumetric integral

$$V_t = \int_{V_t} \mathrm{d}V_t = \int_{V_0} J\,\mathrm{d}V_0 > 0$$

Hence

$$\boxed{J = \det|F| > 0} \quad \text{in } \Omega_0$$

Approximation

$$V_0 \rightarrow 0 : \quad \int_{V_0} J\,\mathrm{d}V_0 \simeq J V_0 \quad \Rightarrow \quad J \simeq V_t/V_0$$

Example 1: Domain mapping (1/2)

Mapping

$$\left.\begin{aligned} x_1 &= X_1 + at^2 \\ x_2 &= X_2 + bX_2t + ct^2 \end{aligned}\right\} a, b, c \in \mathsf{R}$$

Example 1: Domain mapping (1/2)

Mapping

$$\left.\begin{aligned} x_1 &= X_1 + at^2 \\ x_2 &= X_2 + bX_2t + ct^2 \end{aligned}\right\} a, b, c \in \mathrm{R}$$

Initial condition

$$\begin{aligned} x_1(X_1, X_2, 0) &= X_1 \\ x_2(X_1, X_2, 0) &= X_2 \end{aligned}$$

Example 1: Domain mapping (1/2)

Mapping

$$\left.\begin{aligned} x_1 &= X_1 + at^2 \\ x_2 &= X_2 + bX_2t + ct^2 \end{aligned}\right\} a, b, c \in \mathsf{R}$$

Inverse

$$\begin{aligned} X_1 &= x_1 - at^2 \\ X_2 &= \frac{x_2 - ct^2}{1 + bt} \end{aligned}$$

Initial condition

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Initial condition

$$\begin{aligned} x_1(X_1, X_2, 0) &= X_1 \\ x_2(X_1, X_2, 0) &= X_2 \end{aligned}$$

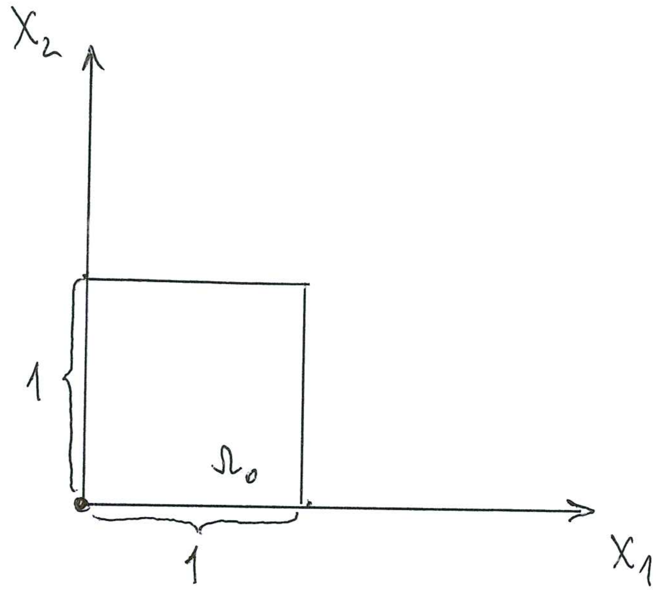
Regularity check

$$b \geq 0: \; t \in [0, \infty)$$
$$b < 0: \; t \in [0, |b|^{-1})$$

Example 1: domain mapping

$$x_1 = X_1 + a t^2$$

$$x_2 = X_2 + b X_2 t + c t^2$$



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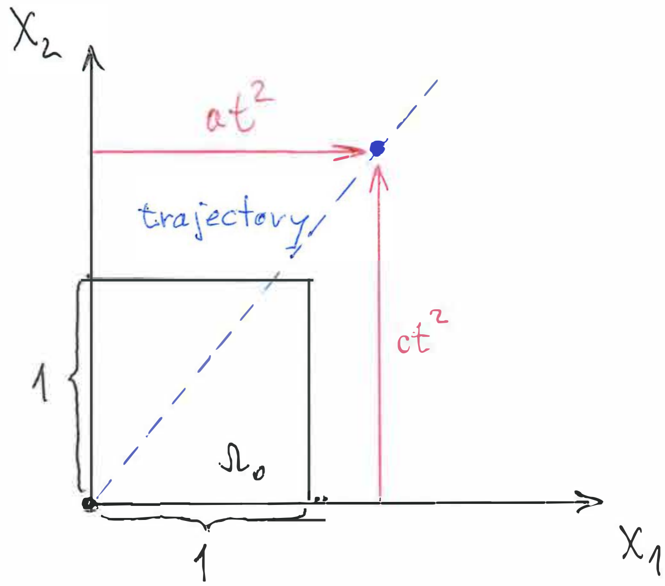

Example 1: domain mapping

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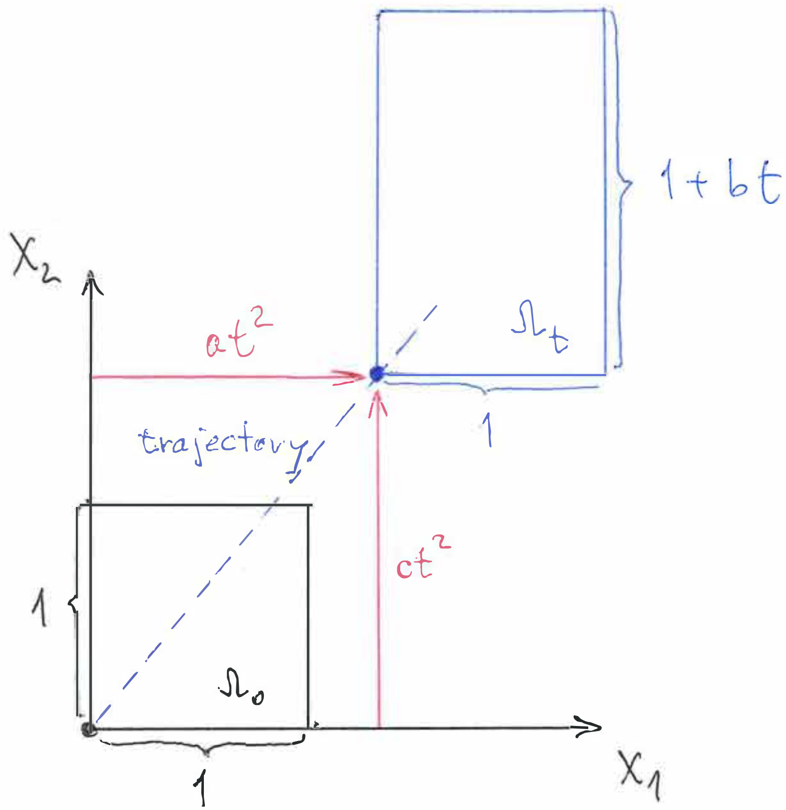


Example 1: Domain mapping (2/2)

Mapping

$$\begin{aligned} x_1 &= X_1 + at^2 \\ x_2 &= X_2 + bX_2t + ct^2 \end{aligned}$$

$$F = \begin{bmatrix} 1 & 0 \\ 0 & 1 + bt \end{bmatrix}$$

Example 1: Domain mapping (2/2)

Mapping

$$
\begin{aligned}
x_1 &= X_1 + at^2 \\
x_2 &= X_2 + bX_2 t + ct^2
\end{aligned}
$$

$$
F = \begin{bmatrix} 1 & 0 \\ 0 & 1 + bt \end{bmatrix}
$$

$$
J = 1 + bt > 0
$$

(same condition as before)

Example 1: Domain mapping (2/2)

Mapping

$$
\begin{aligned}
x_1 &= X_1 + at^2 \\
x_2 &= X_2 + bX_2t + ct^2
\end{aligned}
$$

$$F = \begin{bmatrix} 1 & 0 \\ 0 & 1 + bt \end{bmatrix}$$

$$J = 1 + bt > 0$$

(same condition as before)

Volume change

$$\frac{V_t}{V_0} = \frac{1 + bt}{1 \times 1} = J$$

Inverse mapping

Inverse deformation gradient

$$\mathbf{X}(\mathbf{x},t) : \Omega_t \to \Omega_0 \quad \Rightarrow \quad \mathbf{F}^{-1} \;\mapsto\; F^{-1}_{ij} = \frac{\partial X_i}{\partial x_j}$$

Inverse mapping

Inverse deformation gradient

$$\mathbf{X}(\mathbf{x}, t) : \Omega_t \to \Omega_0 \quad \Rightarrow \quad \mathbf{F}^{-1} \mapsto F_{ij}^{-1} = \frac{\partial X_i}{\partial x_j}$$

Inverse matrix

$$[F][F]^{-1} = [I]$$

Inverse mapping

Inverse deformation gradient

$$\mathbf{X}(\mathbf{x},t) : \Omega_t \to \Omega_0 \quad \Rightarrow \quad \mathbf{F}^{-1} \ \mapsto \ F^{-1}_{ij} = \frac{\partial X_i}{\partial x_j}$$

Inverse matrix

$$[F][F]^{-1} = [I]$$

Proof

$$F_{ik}F^{-1}_{kj} = \frac{\partial x_i}{\partial X_k}\frac{\partial X_k}{\partial x_j} = \frac{\partial x_i}{\partial x_j} = \delta_{ij}$$

Green-Lagrange strain tensor (1/3)

Differential

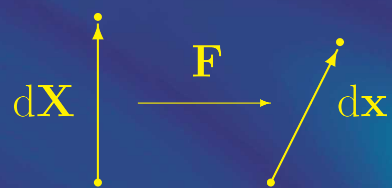
$$\mathrm{d}x_i = \frac{\partial x_i}{\partial X_j}\,\mathrm{d}Xj = F_{ij}\,\mathrm{d}X_j \qquad \Rightarrow \qquad \boxed{\mathrm{d}\{x\} = [F]\,\mathrm{d}\{X\}}$$

Green-Lagrange strain tensor (1/3)

Differential

$$\mathrm{d}x_i = \frac{\partial x_i}{\partial X_j}\,\mathrm{d}Xj = F_{ij}\,\mathrm{d}X_j \qquad \Rightarrow \qquad \boxed{\mathrm{d}\{x\} = [F]\,\mathrm{d}\{X\}}$$

Mapping of a line segment

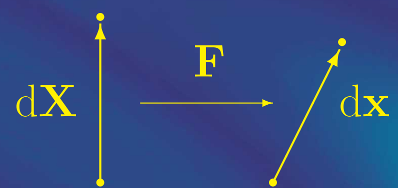


Green-Lagrange strain tensor (1/3)

Differential

$$\mathrm{d}x_i = \frac{\partial x_i}{\partial X_j}\,\mathrm{d}Xj = F_{ij}\,\mathrm{d}X_j \qquad \Rightarrow \qquad \boxed{\mathrm{d}\{x\} = [F]\,\mathrm{d}\{X\}}$$

Mapping of a line segment



denote: $\mathrm{d}L = \|\,\mathrm{d}\mathbf{X}\|$ and $\mathrm{d}l = \|\,\mathrm{d}\mathbf{x}\|$

Green-Lagrange strain tensor (2/3)

Measure of length change

$$
\begin{aligned}
(\mathrm{d}l)^2 - (\mathrm{d}L)^2 &= \|\mathrm{d}\mathbf{x}\|^2 - \|\mathrm{d}\mathbf{X}\|^2 = \mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x} - \mathrm{d}\mathbf{X}\cdot\mathrm{d}\mathbf{X} \\
&= \mathrm{d}\{x\}^T\mathrm{d}\{x\} - \mathrm{d}\{X\}^T\mathrm{d}\{X\}
\end{aligned}
$$

Green-Lagrange strain tensor (2/3)

Measure of length change

$$
\begin{aligned}
(\,\mathrm{d}l)^2 - (\,\mathrm{d}L)^2 &= \|\,\mathrm{d}\mathbf{x}\|^2 - \|\,\mathrm{d}\mathbf{X}\|^2 = \mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x} - \mathrm{d}\mathbf{X}\cdot\mathrm{d}\mathbf{X} \\
&= \mathrm{d}\{x\}^T\,\mathrm{d}\{x\} - \mathrm{d}\{X\}^T\,\mathrm{d}\{X\} \\
&= \mathrm{d}\{X\}^T[F]^T[F]\,\mathrm{d}\{X\} - \mathrm{d}\{X\}^T\,\mathrm{d}\{X\}
\end{aligned}
$$

Green-Lagrange strain tensor (2/3)

Measure of length change

$$\begin{aligned}(\,\mathrm{d}l)^2 - (\,\mathrm{d}L)^2 &= \|\,\mathrm{d}\mathbf{x}\|^2 - \|\,\mathrm{d}\mathbf{X}\|^2 = \mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x} - \mathrm{d}\mathbf{X}\cdot\mathrm{d}\mathbf{X}\\ &= \mathrm{d}\{x\}^T\,\mathrm{d}\{x\} - \mathrm{d}\{X\}^T\,\mathrm{d}\{X\}\\ &= \mathrm{d}\{X\}^T[F]^T[F]\,\mathrm{d}\{X\} - \mathrm{d}\{X\}^T\,\mathrm{d}\{X\}\\ &= \mathrm{d}\{X\}^T\left([F]^T[F] - [I]\right)\mathrm{d}\{X\}\end{aligned}$$

Green-Lagrange strain tensor (2/3)

Measure of length change

$$
\begin{aligned}
(\,\mathrm{d}l)^2 - (\,\mathrm{d}L)^2 &= \|\,\mathrm{d}\mathbf{x}\|^2 - \|\,\mathrm{d}\mathbf{X}\|^2 = \mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x} - \mathrm{d}\mathbf{X}\cdot\mathrm{d}\mathbf{X} \\
&= \mathrm{d}\{x\}^T\,\mathrm{d}\{x\} - \mathrm{d}\{X\}^T\,\mathrm{d}\{X\} \\
&= \mathrm{d}\{X\}^T[F]^T[F]\,\mathrm{d}\{X\} - \mathrm{d}\{X\}^T\,\mathrm{d}\{X\} \\
&= \mathrm{d}\{X\}^T\,\underbrace{([F]^T[F]-[I])}_{2[e]}\,\mathrm{d}\{X\}
\end{aligned}
$$

Component definition

$$
\boxed{\mathbf{e}:\ [e] = \frac{1}{2}(\,[F]^T[F]-[I]\,)}
$$

sym

Green-Lagrange strain tensor (2/3)

Measure of length change

$$\begin{aligned}
(\mathrm{d}l)^2 - (\mathrm{d}L)^2 &= \|\mathrm{d}\mathbf{x}\|^2 - \|\mathrm{d}\mathbf{X}\|^2 = \mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x} - \mathrm{d}\mathbf{X}\cdot\mathrm{d}\mathbf{X} \\
&= \mathrm{d}\{x\}^T\mathrm{d}\{x\} - \mathrm{d}\{X\}^T\mathrm{d}\{X\} \\
&= \mathrm{d}\{X\}^T[F]^T[F]\,\mathrm{d}\{X\} - \mathrm{d}\{X\}^T\mathrm{d}\{X\} \\
&= \mathrm{d}\{X\}^T\underbrace{\left([F]^T[F]-[I]\right)}_{2[e]}\mathrm{d}\{X\}
\end{aligned}$$

Direct notation

$$\boxed{\mathbf{e} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})}$$

sym

Green-Lagrange strain tensor (3/3)

Theorem:

$$\boxed{\mathbf{e} = \mathbf{0} \quad \Longleftrightarrow \quad \forall\, \mathrm{d}\mathbf{X} : \;\; \mathrm{d}L = \mathrm{d}l}$$

Green-Lagrange strain tensor (3/3)

Theorem: $$\boxed{\mathbf{e} = \mathbf{0} \iff \forall\, \mathrm{d}\mathbf{X} : \; \mathrm{d}L = \mathrm{d}l}$$

General proof for zero quadratics

$\{x\}^T[U]\{x\} = 0$ for $\forall\{x\}$

Green-Lagrange strain tensor (3/3)

Theorem: $$\boxed{\mathbf{e} = \mathbf{0} \quad \Longleftrightarrow \quad \forall\, \mathrm{d}\mathbf{X} : \;\; \mathrm{d}L = \mathrm{d}l}$$

General proof for zero quadratics

$\{x\}^T[U]\{x\} = \{x\}^T[\Phi][\Lambda][\Phi]^T\{x\} = 0 \quad$ for $\forall\{x\}$

Green-Lagrange strain tensor (3/3)

Theorem: $\boxed{\mathbf{e} = \mathbf{0} \iff \forall\, \mathrm{d}\mathbf{X} : \ \mathrm{d}L = \mathrm{d}l}$

General proof for zero quadratics

$$\{x\}^T[U]\{x\} = \{x\}^T[\Phi][\Lambda]\underbrace{[\Phi]^T\{x\}}_{\{y\}} = 0 \quad \text{for } \forall\{x\}$$

Green-Lagrange strain tensor (3/3)

Theorem: $$\boxed{\mathbf{e} = \mathbf{0} \iff \forall\, \mathrm{d}\mathbf{X} : \ \mathrm{d}L = \mathrm{d}l}$$

General proof for zero quadratics

$$\{x\}^T[U]\{x\} = \{x\}^T[\Phi][\Lambda]\underbrace{[\Phi]^T\{x\}}_{\{y\}} = \{y\}^T[\Lambda]\{y\} = 0 \quad \text{for } \forall\{x\}$$

Green-Lagrange strain tensor (3/3)

Theorem: $\boxed{\mathbf{e} = \mathbf{0} \iff \forall\, \mathrm{d}\mathbf{X} :\ \mathrm{d}L = \mathrm{d}l}$

General proof for zero quadratics

$$\{x\}^T[U]\{x\} = \{x\}^T[\Phi][\Lambda]\underbrace{[\Phi]^T\{x\}}_{\{y\}} = \{y\}^T[\Lambda]\{y\} = 0 \quad \text{for } \forall\{x\}$$

Multiplying diagonal

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 = 0 \quad \text{for } \forall\{y\} \quad \Rightarrow \quad \lambda_i = 0 \quad \Rightarrow \quad [U] = [0]$$

Green-Lagrange strain tensor (3/3)

Theorem: $$\boxed{\mathbf{e} = \mathbf{0} \quad \Longleftrightarrow \quad \forall\, \mathrm{d}\mathbf{X} : \ \ \mathrm{d}L = \mathrm{d}l}$$

General proof for zero quadratics

$$\{x\}^T[U]\{x\} = \{x\}^T[\Phi][\Lambda]\underbrace{[\Phi]^T\{x\}}_{\{y\}} = \{y\}^T[\Lambda]\{y\} = 0 \qquad \text{for } \forall\{x\}$$

Multiplying diagonal

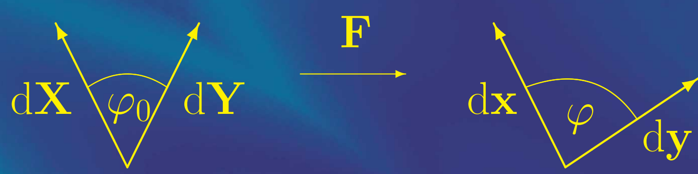
$$\lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 = 0 \qquad \text{for } \forall\{y\} \quad \Rightarrow \quad \lambda_i = 0 \quad \Rightarrow \quad [U] = [0]$$

However

$$[\Phi]^T\{x\} = \{y\}\ , \quad [\Phi] = \text{regular} \quad \Rightarrow \quad \{y\} \mapsto \{x\}$$

Angular distortion

Angular distortion



Bilinear form

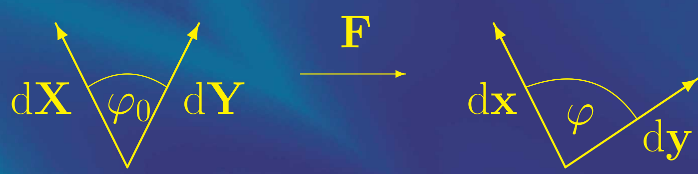
$$\mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{y} - \mathrm{d}\mathbf{X}\cdot\mathrm{d}\mathbf{Y} = \mathrm{d}\{x\}^T\mathrm{d}\{y\} - \mathrm{d}\{X\}^T\mathrm{d}\{Y\}$$

Angular distortion

Bilinear form

$$
\begin{aligned}
\mathrm{d}\mathbf{x} \cdot \mathrm{d}\mathbf{y} - \mathrm{d}\mathbf{X} \cdot \mathrm{d}\mathbf{Y} &= \mathrm{d}\{x\}^T \mathrm{d}\{y\} - \mathrm{d}\{X\}^T \mathrm{d}\{Y\} \\
&= \mathrm{d}\{X\}^T [F]^T [F]\, \mathrm{d}\{Y\} - \mathrm{d}\{X\}^T \mathrm{d}\{Y\}
\end{aligned}
$$

Angular distortion



Bilinear form

$$
\begin{aligned}
\mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{y} - \mathrm{d}\mathbf{X}\cdot\mathrm{d}\mathbf{Y} &= \mathrm{d}\{x\}^T\,\mathrm{d}\{y\} - \mathrm{d}\{X\}^T\,\mathrm{d}\{Y\}\\
&= \mathrm{d}\{X\}^T[F]^T[F]\,\mathrm{d}\{Y\} - \mathrm{d}\{X\}^T\,\mathrm{d}\{Y\}\\
&= \mathrm{d}\{X\}^T\,\underbrace{\left([F]^T[F]-[I]\right)}_{2[e]}\,\mathrm{d}\{Y\}
\end{aligned}
$$

Angular distortion

Bilinear form

$$
\begin{aligned}
\mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{y}-\mathrm{d}\mathbf{X}\cdot\mathrm{d}\mathbf{Y} &= \mathrm{d}\{x\}^T\mathrm{d}\{y\}-\mathrm{d}\{X\}^T\mathrm{d}\{Y\}\\
&= \mathrm{d}\{X\}^T[F]^T[F]\,\mathrm{d}\{Y\}-\mathrm{d}\{X\}^T\mathrm{d}\{Y\}\\
&= \mathrm{d}\{X\}^T\underbrace{\left([F]^T[F]-[I]\right)}_{2[e]}\mathrm{d}\{Y\}
\end{aligned}
$$

corollary: $\boxed{\mathbf{e}=\mathbf{0}\quad\Rightarrow\quad \mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{y}=\mathrm{d}\mathbf{X}\cdot\mathrm{d}\mathbf{Y}}$ for all directions

Non-uniqueness

Scalar product

$$\mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{y} \;=\; \|\mathrm{d}\mathbf{x}\|\|\mathrm{d}\mathbf{y}\|\cos\varphi$$

Non-uniqueness

Scalar product

$$\mathrm{d}\mathbf{x} \cdot \mathrm{d}\mathbf{y} \;=\; \|\,\mathrm{d}\mathbf{x}\|\|\,\mathrm{d}\mathbf{y}\|\cos\varphi \;=\; \|\,\mathrm{d}\mathbf{X}\|\|\,\mathrm{d}\mathbf{Y}\|\cos\varphi$$

Non-uniqueness

Scalar product

$$\mathrm{d}\mathbf{x} \cdot \mathrm{d}\mathbf{y} = \|\mathrm{d}\mathbf{x}\| \|\mathrm{d}\mathbf{y}\| \cos\varphi = \|\mathrm{d}\mathbf{X}\| \|\mathrm{d}\mathbf{Y}\| \cos\varphi$$

$$\mathrm{d}\mathbf{X} \cdot \mathrm{d}\mathbf{Y} = \|\mathrm{d}\mathbf{X}\| \|\mathrm{d}\mathbf{Y}\| \cos\varphi_0$$

Non-uniqueness

Scalar product

$$\begin{aligned} \mathrm{d}\mathbf{x} \cdot \mathrm{d}\mathbf{y} &= \|\mathrm{d}\mathbf{x}\| \|\mathrm{d}\mathbf{y}\| \cos\varphi = \|\mathrm{d}\mathbf{X}\| \|\mathrm{d}\mathbf{Y}\| \cos\varphi \\ \mathrm{d}\mathbf{X} \cdot \mathrm{d}\mathbf{Y} &= \|\mathrm{d}\mathbf{X}\| \|\mathrm{d}\mathbf{Y}\| \cos\varphi_0 \end{aligned}$$

Comparing: $\cos\varphi = \cos\varphi_0$

Non-uniqueness

Scalar product

$$\begin{aligned} \mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{y} &= \|\,\mathrm{d}\mathbf{x}\|\|\,\mathrm{d}\mathbf{y}\|\cos\varphi = \|\,\mathrm{d}\mathbf{X}\|\|\,\mathrm{d}\mathbf{Y}\|\cos\varphi \\ \mathrm{d}\mathbf{X}\cdot\mathrm{d}\mathbf{Y} &= \|\,\mathrm{d}\mathbf{X}\|\|\,\mathrm{d}\mathbf{Y}\|\cos\varphi_0 \end{aligned}$$

Comparing: $\cos\varphi = \cos\varphi_0 \;\Rightarrow\; |\varphi| = |\varphi_0|$

Non-uniqueness

Scalar product

$$\mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{y} = \|\mathrm{d}\mathbf{x}\|\|\mathrm{d}\mathbf{y}\|\cos\varphi = \|\mathrm{d}\mathbf{X}\|\|\mathrm{d}\mathbf{Y}\|\cos\varphi$$

$$\mathrm{d}\mathbf{X}\cdot\mathrm{d}\mathbf{Y} = \|\mathrm{d}\mathbf{X}\|\|\mathrm{d}\mathbf{Y}\|\cos\varphi_0$$

Comparing: $\cos\varphi = \cos\varphi_0 \Rightarrow |\varphi| = |\varphi_0|$

Two solutions

Non-uniqueness

Scalar product

$$
\begin{aligned}
\mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{y} &= \|\,\mathrm{d}\mathbf{x}\|\|\,\mathrm{d}\mathbf{y}\|\cos\varphi = \|\,\mathrm{d}\mathbf{X}\|\|\,\mathrm{d}\mathbf{Y}\|\cos\varphi \\
\mathrm{d}\mathbf{X}\cdot\mathrm{d}\mathbf{Y} &= \|\,\mathrm{d}\mathbf{X}\|\|\,\mathrm{d}\mathbf{Y}\|\cos\varphi_0
\end{aligned}
$$

Comparing: $\cos\varphi = \cos\varphi_0 \;\Rightarrow\; |\varphi| = |\varphi_0|$

Two solutions

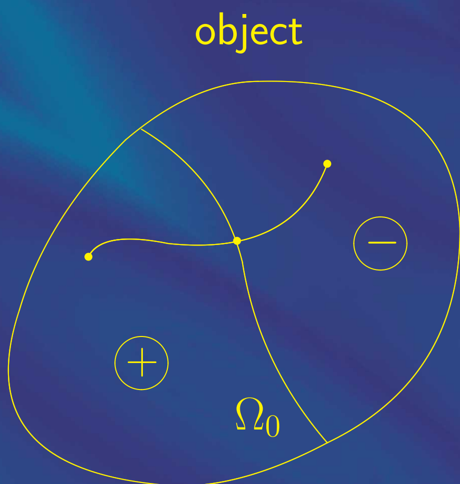
$J = +1$ $\qquad$ $J = -1$

Mirroring

$\mathbf{F} = \mathbf{I}$

$J = 1$

$\mathbf{e} = \mathbf{0}$

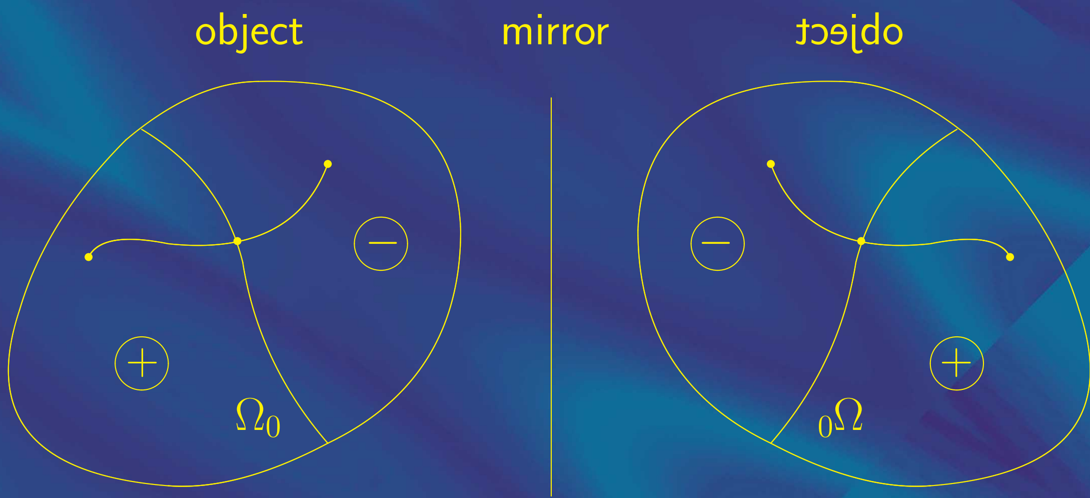


Mirroring

$\mathbf{F} = \mathbf{I}$

$J = 1$

$\mathbf{e} = \mathbf{0}$



$\mathbf{F} = -\mathbf{I}$

$J = -1$

$\mathbf{e} = \mathbf{0}$