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MECHANICS OF FLUIDS

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- Constitutive equations
- Inviscid fluids
  Ideal gas
- Navier-Stokes equations
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Euler description

Five conditions to meet:

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Five equations to solve: v_1, v_2, v_3, T, ρ (or pressure).

Constitutive equations

Kinematics

$$\mathbf{v} \quad \mapsto \quad \mathbf{L} = \mathbf{D} + \mathbf{W}$$

Constitutive equations satisfying 3. and 5.

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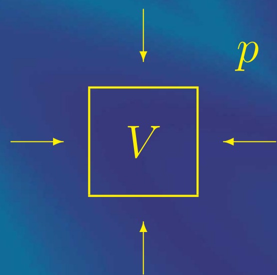
Constitutive equations satisfying 3. and 5.

at time t: input $\rho(\mathbf{x}, t)$, $T(\mathbf{x}, t)$, $\mathbf{D}(\mathbf{x}, t)$ and compute

- $\boldsymbol{\sigma}(\mathbf{x}, t) = -p\mathbf{I} + \boldsymbol{\tau}$
- $\mathbf{h}(\mathbf{x}, t)$
- $\psi(\mathbf{x}, t) \quad \mapsto \quad \eta, \ u$

Inviscid fluids (1/4)

Assumption



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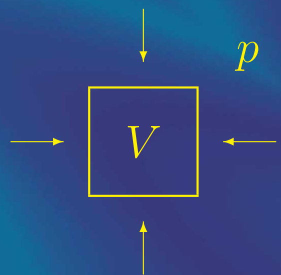

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$$\frac{\partial \sigma_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}(-p\delta_{ij}) = -\frac{\partial p}{\partial x_i}$$

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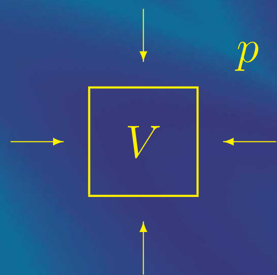

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$$\boxed{\operatorname{div} \boldsymbol{\sigma} = -\operatorname{grad} p}$$

Inviscid fluids (2/4)

Power density

$$\boldsymbol{\sigma} : \mathbf{D} = \sigma_{ij} D_{ij} = -p\delta_{ij} D_{ij} = -p D_{ii}$$

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Continuity equation

$$-\operatorname{div} \mathbf{v} = \frac{\dot{\rho}}{\rho}$$

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Continuity equation

$$-\mathrm{div}\,\mathbf{v} = \frac{\dot{\rho}}{\rho}$$

$$\boxed{\boldsymbol{\sigma} : \mathbf{D} = p\,\frac{\dot{\rho}}{\rho}}$$

Inviscid fluids (3/4)

1. $\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0$

2. $-\operatorname{grad} p + \mathbf{b} = \rho \dot{\mathbf{v}}$ (Euler)

4. $\kappa - \operatorname{div} \mathbf{h} + p \dfrac{\dot{\rho}}{\rho} = \rho \dot{u}$

5. $-\rho \eta \dot{T} + p \dfrac{\dot{\rho}}{\rho} - \dfrac{1}{T} \mathbf{h} \cdot \operatorname{grad} T \geq \rho \dot{\psi}$

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Corollary

$$\boxed{\psi = \text{function}(\rho, T, \cdots)}$$

Inviscid fluids (4/4)

Assumption

$$\exists \psi(\rho, T): \quad \dot{\psi} = \frac{\partial \psi}{\partial \rho}\dot{\rho} + \frac{\partial \psi}{\partial T}\dot{T}$$

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Dissipation inequality

$$-\rho\left(\eta + \frac{\partial \psi}{\partial T}\right)\dot{T} + \left(\frac{p}{\rho} - \rho\,\frac{\partial \psi}{\partial \rho}\right)\dot{\rho} - \frac{1}{T}\mathbf{h}\cdot\operatorname{grad} T \geq 0$$

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$$\boxed{\eta = -\frac{\partial \psi}{\partial T} \quad \text{and} \quad p = \rho^2\,\frac{\partial \psi}{\partial \rho}}$$

Remark: Incompressibility

1. $\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0$

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4. $\kappa - \operatorname{div} \mathbf{h} = \rho_0 \dot{u}$

5. $\mathbf{h} \cdot \operatorname{grad} T \leq 0$

Note that $\dot{u}$ includes convection as

$$\dot{u} = \left(\frac{\partial u}{\partial t} \right)_x + (\operatorname{grad} u) \mathbf{v}$$

Ideal gas (1/2)

Experiments

$p(V)$ Boyle & Mariotte (1662)

$p(T)$ Gay-Lussac (1808)

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Equation of state

$$\frac{p}{\rho} = rT$$

where r [J/kgK] is the specific (not universal) gas constant.

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Helmholtz free energy

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Internal energy

$u = \psi + T\eta$

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Corollary

$$\boxed{u = \text{function}(T)}$$

Compare to the kinetic theory of gases.

Navier-Stokes equations (1/2)

Activating viscous stress, it follows from CDI

$$\boldsymbol{\tau} : \mathbf{D} \geq 0 \quad \Rightarrow \quad \boldsymbol{\tau} = \text{function}(\mathbf{D})$$

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Linear model

$$\boldsymbol{\tau} = \mathbf{CD} \qquad \text{(Newton-Stokes)}$$

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$$\boldsymbol{\tau} = \mathbf{C}\mathbf{D} \qquad \text{(Newton-Stokes)}$$

Isotropy

$$\boldsymbol{\tau} = \lambda \, \mathrm{tr}\,(\mathbf{D})\mathbf{I} + 2\mu\mathbf{D}$$

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$$\operatorname{tr}(\mathbf{D}) = D_{ii} = L_{ii} = \operatorname{div} \mathbf{v}$$

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. . . hence

$$\boxed{\boldsymbol{\tau} = \lambda(\operatorname{div} \mathbf{v})\mathbf{I} + 2\mu\mathbf{D}}$$

Navier-Stokes equations (2/2)

Inserting into the equations of motion

$$\boxed{-\operatorname{grad} p + (\lambda + \mu)\operatorname{grad}(\operatorname{div} \mathbf{v}) + \mu \operatorname{div} \mathbf{L} + \mathbf{b} = \rho \left(\frac{\partial \mathbf{v}}{\partial t}\right)_x + \rho \mathbf{L}\mathbf{v}}$$

μ = dynamic viscosity (Newton)
λ = second viscosity

Navier-Stokes equations (2/2)

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Bulk viscosity

$$\mathrm{tr}\,(\boldsymbol{\tau}) = \underbrace{(3\lambda + 2\mu)}_{3K}\,\mathrm{div}\,\mathbf{v}$$

Navier-Stokes equations (2/2)

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$\mu =$ dynamic viscosity (Newton)
$\lambda =$ second viscosity

Bulk viscosity

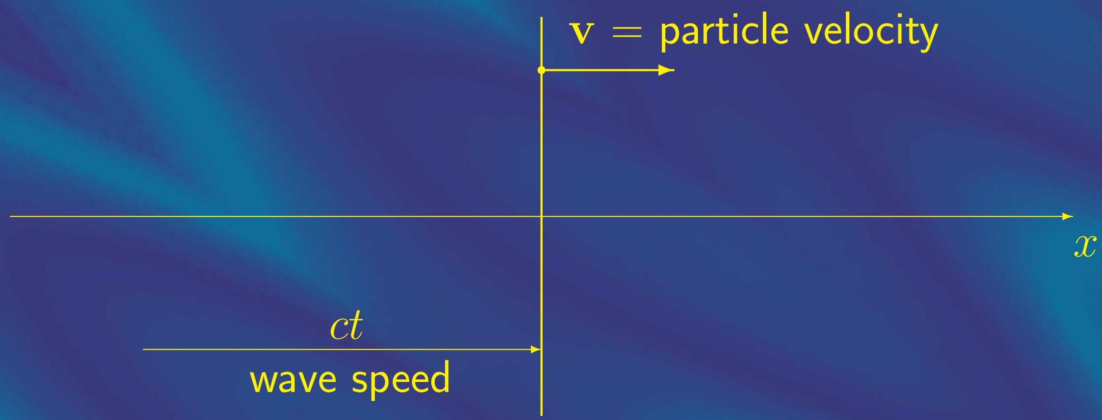
$$\operatorname{tr}(\boldsymbol{\tau}) = \underbrace{(3\lambda+2\mu)}_{3K}\operatorname{div}\mathbf{v}$$

Stokes condition

$$K = \lambda + \frac{2}{3}\mu \equiv \zeta = 0$$

Wave propagation

Hydrodynamic approximation; longitudinal planar wave



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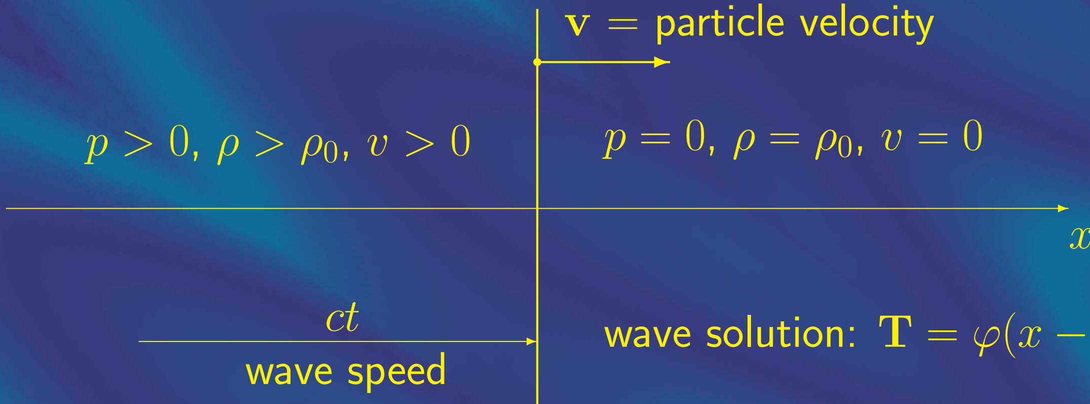

wave solution: $\mathbf{T} = \varphi(x - ct) = \varphi(\xi)$

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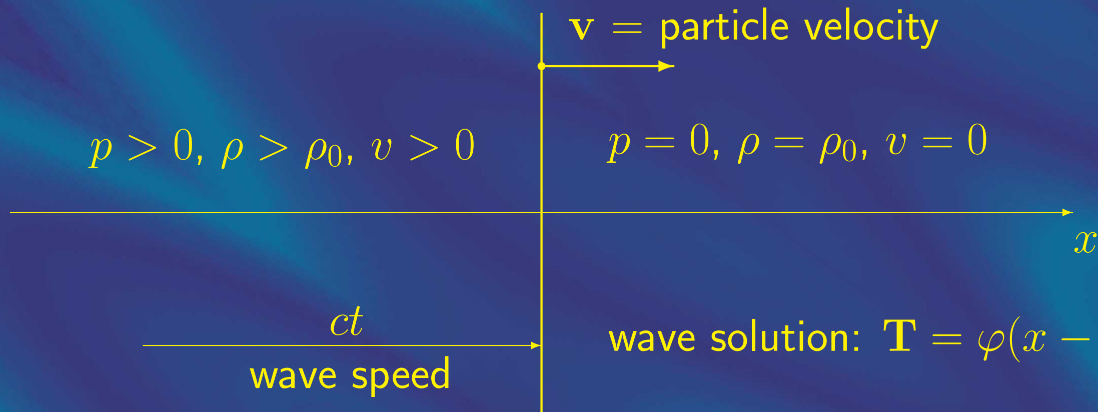

wave solution: $\mathbf{T} = \varphi(x - ct) = \varphi(\xi)$

Property:

$$\left(\frac{\partial \mathbf{T}}{\partial t}\right)_x = \frac{\partial \varphi}{\partial t} = \frac{\mathrm{d}\varphi}{\mathrm{d}\xi}\frac{\partial \xi}{\partial t} = \frac{\mathrm{d}\varphi}{\mathrm{d}\xi}(-c) = -c\,\frac{\partial \varphi}{\partial x} = -c\,\frac{\partial \mathbf{T}}{\partial x}$$

Continuity equation

Spatial description

$$\left(\frac{\partial \rho}{\partial t}\right)_x + \operatorname{div}(\rho \mathbf{v}) = 0$$

Continuity equation

Spatial description

$$\left(\frac{\partial \rho}{\partial t}\right)_x + \operatorname{div}(\rho \mathbf{v}) = 0$$

Substitution

$$-c\,\frac{\partial \rho}{\partial x} + \frac{\partial}{\partial x}(\rho v) = 0$$

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Integration

$$-c\rho + \rho v = f(t)$$

Continuity equation

Spatial description

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$$-c\,\frac{\partial \rho}{\partial x} + \frac{\partial}{\partial x}(\rho v) = 0$$

Integration

$$-c\rho + \rho v = f(t)$$

Initial condition

for $x > ct : -c\rho_0 + 0 = f(t)$

Continuity equation

Spatial description

$$\left(\frac{\partial\rho}{\partial t}\right)_x + \operatorname{div}(\rho\mathbf{v}) = 0$$

Substitution

$$-c\,\frac{\partial\rho}{\partial x} + \frac{\partial}{\partial x}(\rho v) = 0$$

Integration

$$-c\rho + \rho v = f(t)$$

Initial condition

$$\text{for } x > ct : -c\rho_0 + 0 = f(t)$$

Universal solution

$$\boxed{\rho(c - v) = \rho_0 c}$$

Remark: No supersonic speeds!

Equation of motion

Spatial description

$$-\operatorname{grad} p + \mathbf{b} = \rho \left(\frac{\partial \mathbf{v}}{\partial t} \right)_x + \rho \mathbf{L} \mathbf{v}$$

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$$-\frac{\partial p}{\partial x} = -\rho c \frac{\partial v}{\partial x} + \rho \frac{\partial v}{\partial x} v$$

Equation of motion

Spatial description

$$-\operatorname{grad} p + \mathbf{b} = \rho \left(\frac{\partial \mathbf{v}}{\partial t} \right)_x + \rho \mathbf{L} \mathbf{v}$$

Substitution

$$\frac{\partial p}{\partial x} = \rho c \frac{\partial v}{\partial x} - \rho \frac{\partial v}{\partial x} v$$

Equation of motion

Spatial description

$$-\operatorname{grad} p + \mathbf{b} = \rho \left(\frac{\partial \mathbf{v}}{\partial t} \right)_x + \rho \mathbf{L} \mathbf{v}$$

Substitution

$$\frac{\partial p}{\partial x} = \rho (c - v) \frac{\partial v}{\partial x}$$

Equation of motion

Spatial description

$$-\operatorname{grad} p + \mathbf{b} = \rho \left(\frac{\partial \mathbf{v}}{\partial t} \right)_x + \rho \mathbf{L} \mathbf{v}$$

Substitution

$$\frac{\partial p}{\partial x} = \underbrace{\rho (c - v)}_{\rho_0 c} \frac{\partial v}{\partial x}$$

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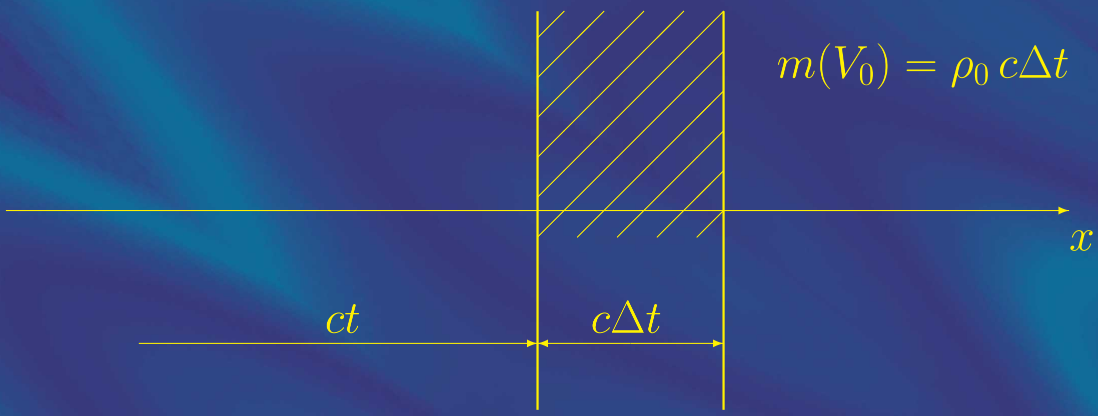
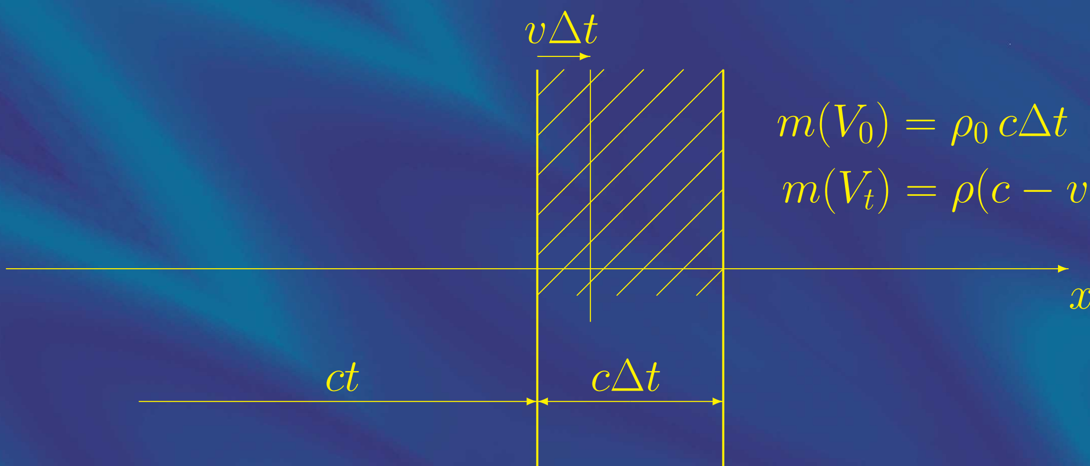
Universal solution

$$\boxed{p = \rho_0 c v}$$

Shock loading



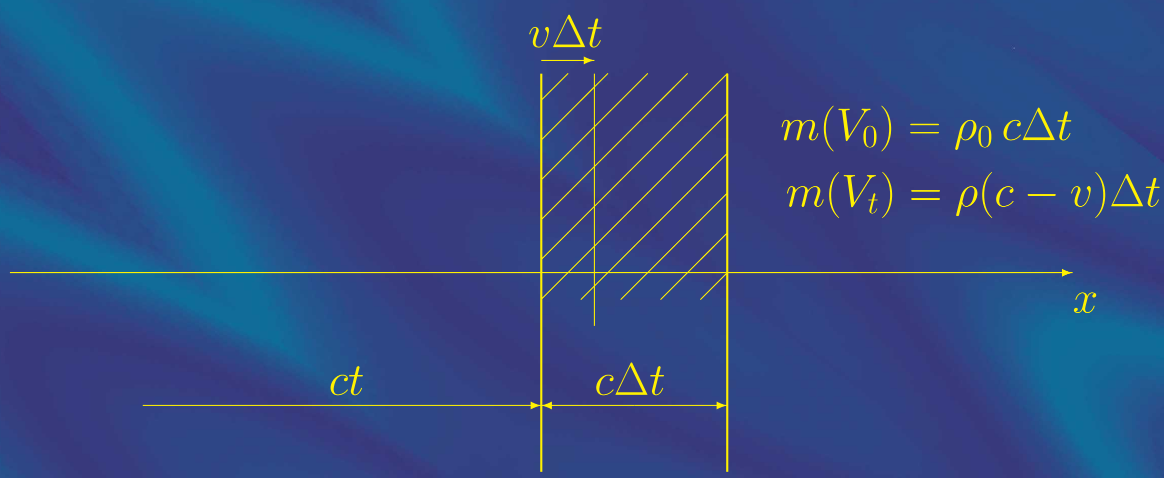
Shock loading



$$m(V_0) = \rho_0\, c\Delta t$$

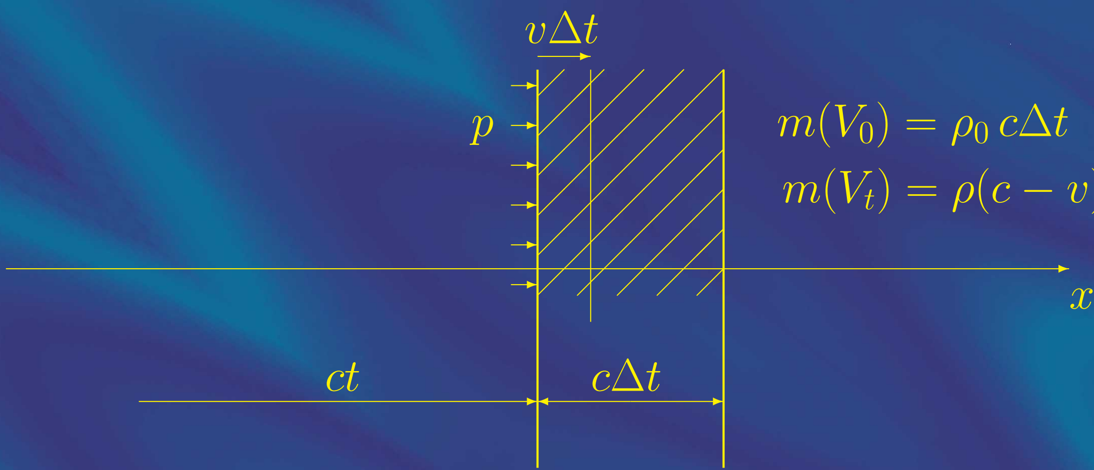
$$m(V_t) = \rho(c - v)\Delta t$$

Shock loading



Mass: $\rho_0\, c\Delta t = \rho(c - v)\Delta t$ (checks)

Shock loading



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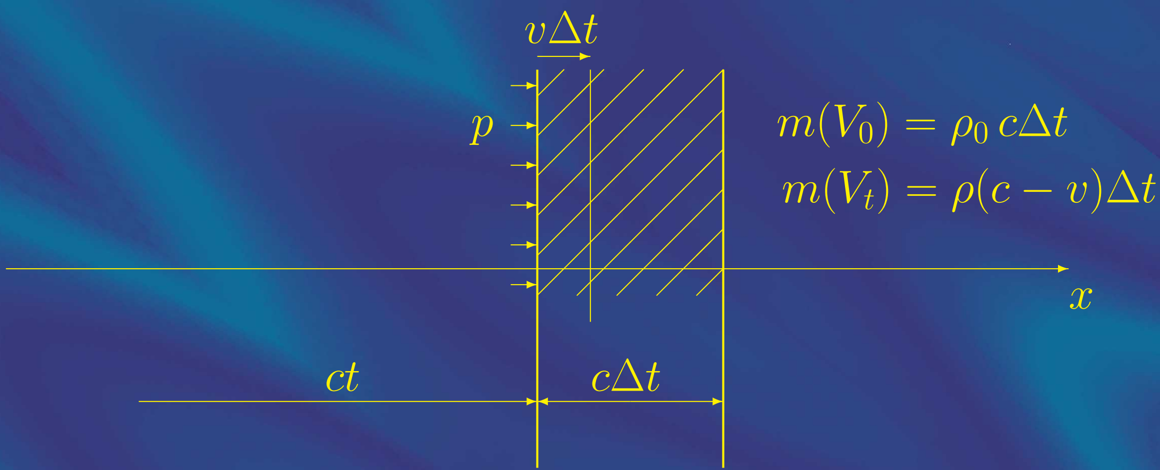
Shock loading



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Note: The previous solution holds valid for any (non-shock) wave profile

Shock equations

Universal solution

$$\rho_0\, c = \rho(c - v)$$

$$p = \rho_0\, c v$$

uknowns: ρ, v, p

Shock equations

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1st eqn.: $v = c(1 - \dfrac{\rho_0}{\rho})$

Shock equations

Universal solution

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$$p = \rho_0\, cv$$

uknowns: ρ, v, p

1st eqn.: $v = c(1 - \dfrac{\rho_0}{\rho}) = c(1 - J)$

Shock equations

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Wave speed

$$\boxed{\rho_0\, c^2 = -\frac{p}{\epsilon}}$$

Remark: Note that $-\epsilon$ represents the compression ratio.

Perfect plasticity

Uniaxial strain state

$$-\frac{p}{\epsilon} = K = \frac{E}{3(1-2\nu)}$$

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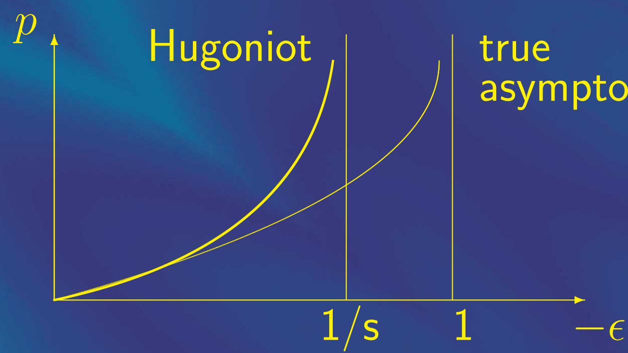
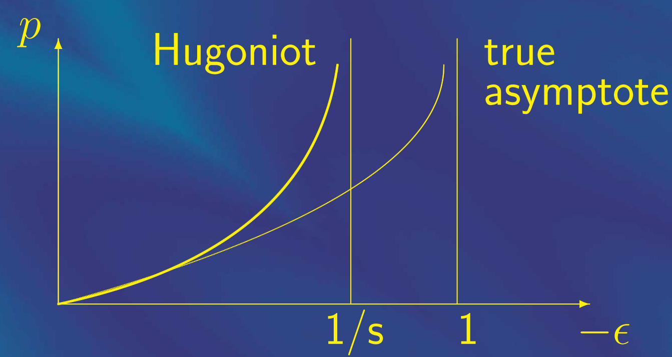
Hugoniot curve



Hugoniot curve



Example: steel

$$K = \frac{E}{1.2} = 167 \text{ GPa} \quad \Rightarrow \quad c_0 = \sqrt{\frac{167 \times 10^9}{7800}} = 4627 \text{ m/s}$$

$$v = 2 \text{ km/s} \quad \Rightarrow \quad c = 7627 \text{ m/s}$$

$$p = 7800 \times 7627 \times 2000 = 120 \text{ GPa}$$