Institute of Thermomechanics ASCR

OBJECTIVE DERIVATIVES

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Contents

Material models of small strain theory

Hyperelastic

$$\sigma_{ij} = \frac{\partial \psi}{\partial \epsilon_{ij}} \quad \Rightarrow \quad \mathrm{d}\sigma_{ij} = \frac{\partial^2 \psi}{\partial \epsilon_{ij} \epsilon_{kl}} \mathrm{d}\epsilon_{kl} = C_{ijkl} \, \mathrm{d}\epsilon_{kl}$$

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$$\boxed{\dot{\boldsymbol{\sigma}} = \mathbf{C}^{\mathrm{hp}} \dot{\boldsymbol{\epsilon}}}$$

Remark: Internal friction, backstress evolution, etc.

Concept of finite strains

Generalization

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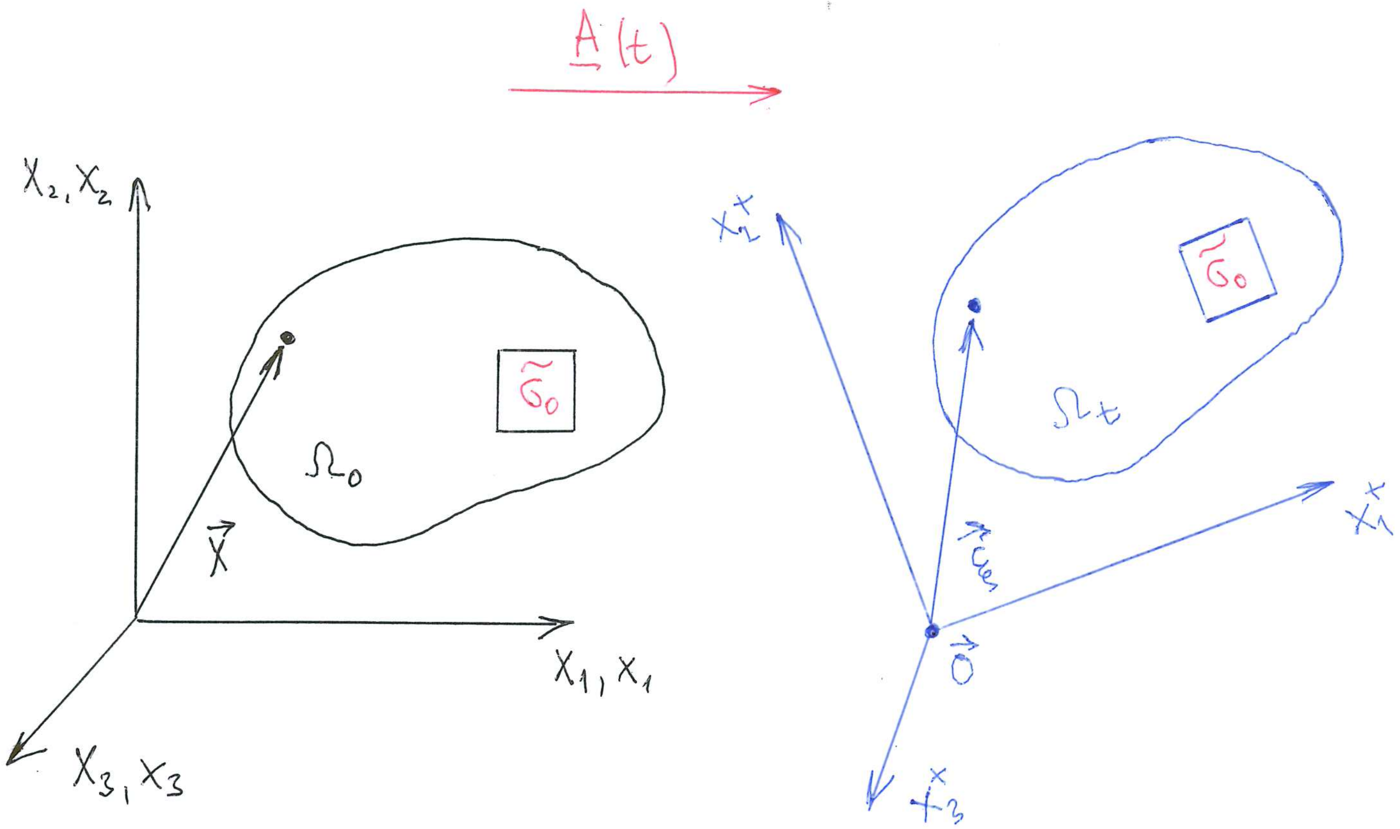
Definition

$$\boxed{\overset{\circ}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \text{ kinematic RBM correction}}$$

Remark: The operation $\overset{\circ}{\boldsymbol{\sigma}}$ is known as the objective (time) derivative.

Rigid body motion



$$\underline{\sigma}^+(t) = \underline{\sigma}_0 \qquad \underline{\xi}^+ = \underline{X} \qquad \vec{x} = \vec{O} + \vec{\xi}$$

Rigid body motion (1/2)

Cauchy stress

$[\sigma] = [A]^T[\sigma^+][A] = [A]^T[\sigma_0][A]$

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Current position vector

$$\{x\} = [A]^T\{x^+\}$$

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$$\mathrm{d}\{x\} = [A]^T\,\mathrm{d}\{X\} \quad \Rightarrow \quad [F] = [A]^T$$

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Hence

$$\boxed{[\dot{\sigma}] = [L][\sigma] + [\sigma][L]^T}$$

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Oldroyd (contravariant)

$$\dot{\boldsymbol{\sigma}} = \mathbf{L}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{L}^T \quad \text{(sym)}$$

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$[D] = \frac{1}{2}([L] + [L]^T)$

$= \frac{1}{2}([\dot{A}]^T[A] + [A]^T[\dot{A}])$

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$$\boxed{\dot{\boldsymbol{\sigma}} = \mathbf{W}\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{W}} \quad \text{(sym)}$$

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$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{R} \;\Rightarrow\; \mathbf{L} = \dot{\mathbf{R}}\mathbf{R}^T = \boldsymbol{\Omega} \equiv \mathbf{W}$$

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Note, that GN-rate depends on $\mathbf{R}$, whereas the previous corrections are solely functions of $\mathbf{L}$.

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. . . and many more, e.g. Truesdell, log-spin, mixed Oldroyd (nonsym.), etc.

Zaremba-Jaumann-Noll rate (1/2)

Assume

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Zaremba(1903), Jaumman (1911), Noll(1940)

$$\boxed{\overset{\circ}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \mathbf{W}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{W}} \qquad \text{(sym)}$$

Zaremba-Jaumann-Noll rate (2/2)

Algorithm:

1. $\mathbf{L} = \mathbf{D} + \mathbf{W}$
2. $\overset{\circ}{\boldsymbol{\sigma}} = \mathbf{C}^{\mathrm{hp}}\mathbf{D}$ (constitutive equations)
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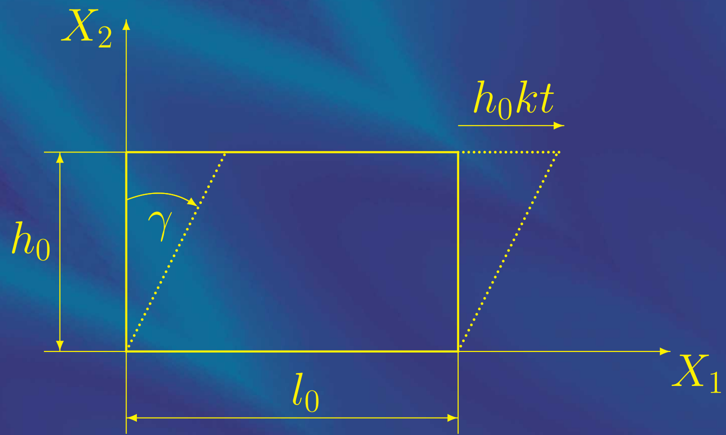
Remarks

- Derivation of ZJN derivative is rigorous.
- ZJN based CE are correct and useful.
- Oldroyd derivatives are equivalent.
- GN derivative is not equivalent as it depends on the reference configuration.

Example 3: Dienes (1979)



$$u_1 = X_2 \tan\gamma$$

$$u_2 = 0$$

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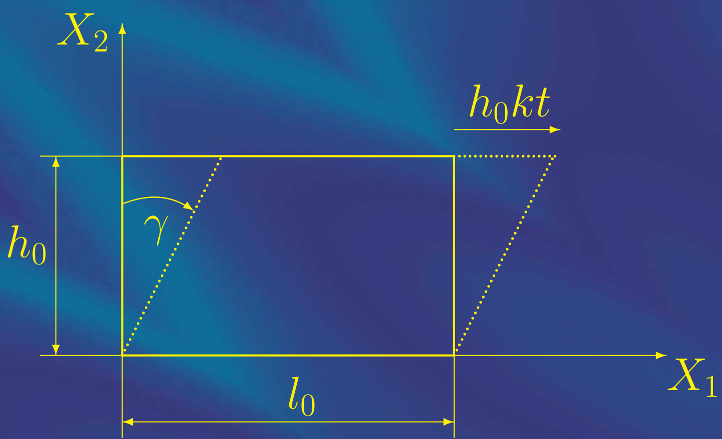

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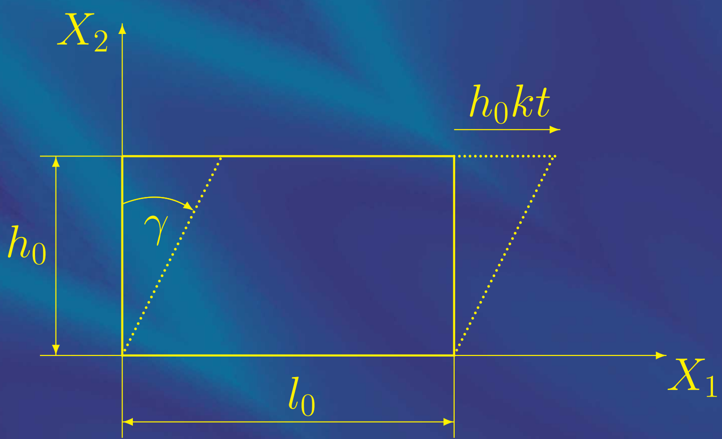

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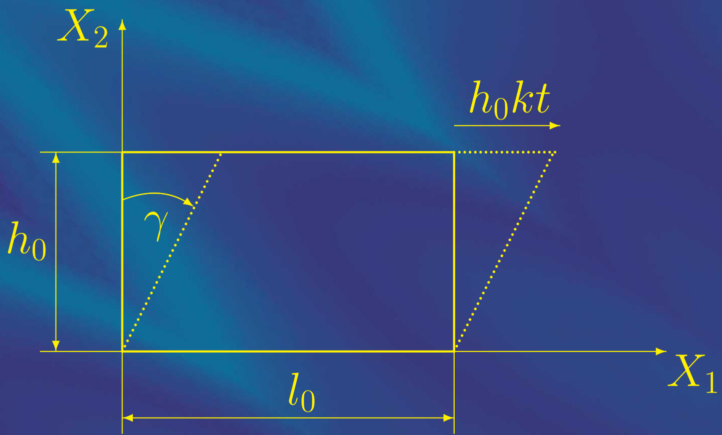

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Universal solution for simple shear

$$[L] = \begin{bmatrix} 0 & k \\ 0 & 0 \end{bmatrix} \qquad [D] = \frac{k}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad [W] = \frac{k}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

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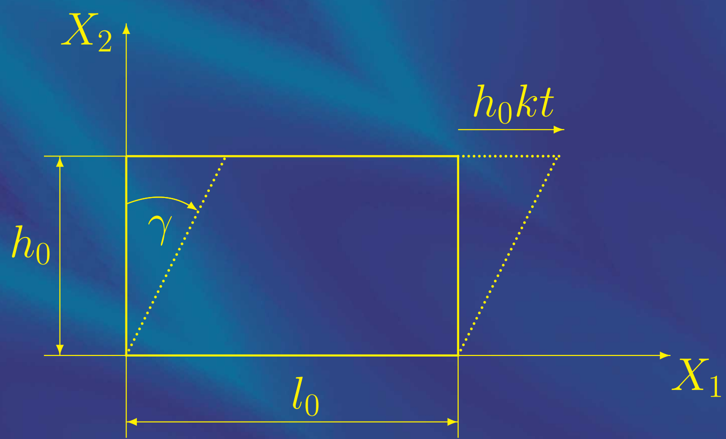

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$$[W][\sigma] - [\sigma][W] = \frac{k}{2}\begin{bmatrix} 2\sigma_{21} & \sigma_{22} - \sigma_{11} \\ \sigma_{22} - \sigma_{11} & -2\sigma_{12} \end{bmatrix} \quad \text{(sym)}$$

Example 3: Dienes (1979)

Hypoelastic eqn. of degree zero with Lamé's constants

$$\boxed{\overset{\circ}{\boldsymbol{\sigma}} = \lambda \,\mathrm{tr}\,(\mathbf{D})\mathbf{I} + 2\mu\mathbf{D}}$$

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Adding ZJN correction

$$\begin{bmatrix} \dot{\sigma}_{11} & \dot{\sigma}_{12} \\ \dot{\sigma}_{21} & \dot{\sigma}_{22} \end{bmatrix} = \mu k \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \frac{k}{2} \begin{bmatrix} 2\sigma_{21} & \sigma_{22} - \sigma_{11} \\ \sigma_{22} - \sigma_{11} & -2\sigma_{12} \end{bmatrix}$$

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Only the specification of objective rate completes the CE. Here we chose ZJN.

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System of three ODE to solve:

$$\dot{\sigma}_{11} = k\sigma_{21}\,, \quad \dot{\sigma}_{22} = -k\sigma_{12}\,, \quad \dot{\sigma}_{12} = \mu k + \frac{k}{2}(\sigma_{22} - \sigma_{11})$$

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Initial condition at $t = 0$: $\sigma_{11} + \sigma_{22} = 0$

Example 3: Dienes (1979)

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Initial condition at $t = 0$: $\sigma_{22} = -\mu + B = 0 \quad \Rightarrow \quad \sigma_{22} = \mu(-1 + \cos kt)$

Example 3: Dienes (1979)

ZJN solution

$$
\begin{aligned}
\sigma_{11} &= -\sigma_{22} = \mu(1 - \cos kt) \\
\sigma_{12} &= \mu \sin kt
\end{aligned}
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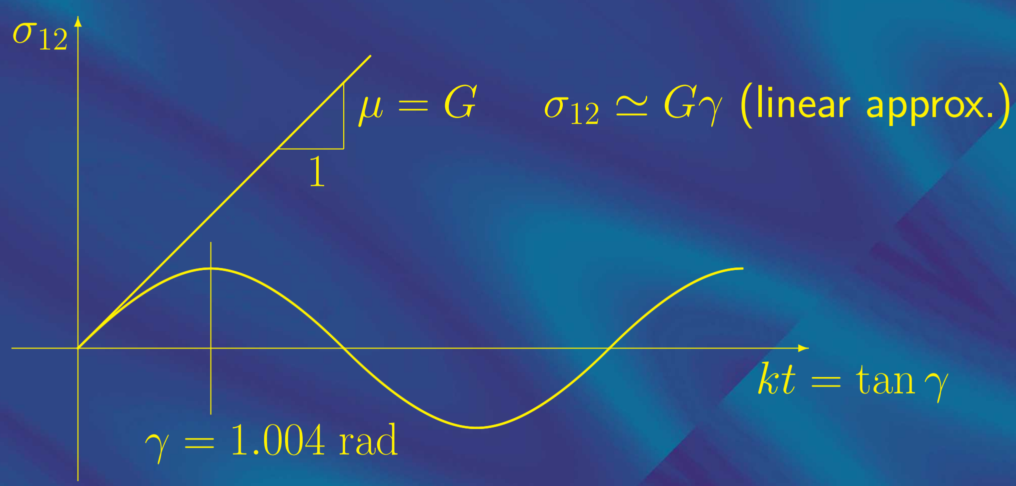

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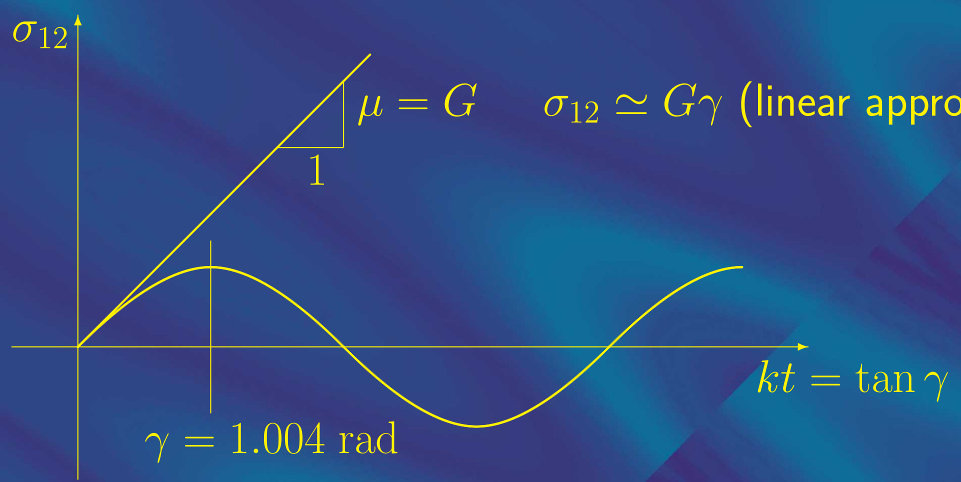

Only valid for reasonable shear angle, i.e., good for most engineering problems.

Conclusions

- Dienes (1979) proposed the use of GN-rate, which was later disputed by Halleux and Donea (1986).
- In fact, Simo and Pister (1984) proved that no commonly known stress rate enables to recover reversible elastic response from the hypoelastic equation of degree zero.
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- ZJN rate is required as an output parameter of the UMAT procedure, which poses less of a problem than often thought.