Institute of Thermomechanics ASCR

CONSERVATION LAWS III

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Contents

- Conservation of energy
  - Conjugate stress
  - Material description
- Clausius-Duhem inequality
  - Local forms
  - Discussion of non-physical solutions
  - Dissipation inequality
- Discussion of the 2nd law
  (optional)

4. Conservation of energy (continued)

Review

$$\dot{Q} + \dot{W} = \frac{\mathrm{d}}{\mathrm{d}t}(U + K)$$

Review

$$\dot{Q} + \dot{W} = \frac{\mathrm{d}}{\mathrm{d}t}(U + K)$$

$$\dot{W} = \frac{\mathrm{d}}{\mathrm{d}t}K + \int_{V_t} \boldsymbol{\sigma} : \mathbf{D} \,\mathrm{d}V_t$$

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Review

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Spatial description

$$\kappa - \operatorname{div}\mathbf{h} + \boldsymbol{\sigma} : \mathbf{D} = \rho\dot{u} \quad \text{in} \quad \Omega$$

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Spatial description

$$\kappa - \operatorname{div} \mathbf{h} + \boldsymbol{\sigma} : \mathbf{D} = \rho \dot{u} \quad \text{in} \quad \Omega$$

Material description

$$J\kappa - J\operatorname{div} \mathbf{h} + J\boldsymbol{\sigma} : \mathbf{D} = J\rho \dot{u} \quad \text{in} \quad \Omega_0$$

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$$\dot{Q} + \dot{W} = \frac{\mathrm{d}}{\mathrm{d}t}(U + K)$$

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$$\dot{Q} + \dot{W} = \frac{\mathrm{d}}{\mathrm{d}t}(U + K)$$

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$$\kappa - \mathrm{div}\,\mathbf{h} + \boldsymbol{\sigma} : \mathbf{D} = \rho \dot{u} \quad \text{in} \quad \Omega$$

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$$\underbrace{J\kappa}_{\mathcal{K}} - \underbrace{J\mathrm{div}\,\mathbf{h}}_{\mathrm{Div}\,\mathbf{H}} + \underbrace{J\boldsymbol{\sigma} : \mathbf{D}}_{\boldsymbol{\Sigma} : \dot{\mathbf{E}}} = \underbrace{J\rho}_{\rho_0}\, \dot{u} \quad \text{in} \quad \Omega_0$$

Piola heat flux vector

Using the Nanson formula . . .

$$\int_{\Delta S_t} \mathbf{h} \cdot \mathbf{n} \, \mathrm{d}S_t = \int_{\Delta S_0} J \mathbf{h} \cdot (\mathbf{F}^{-T} \mathbf{N}) \, \mathrm{d}S_0$$

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Piola heat flux vector

Using the Nanson formula ...

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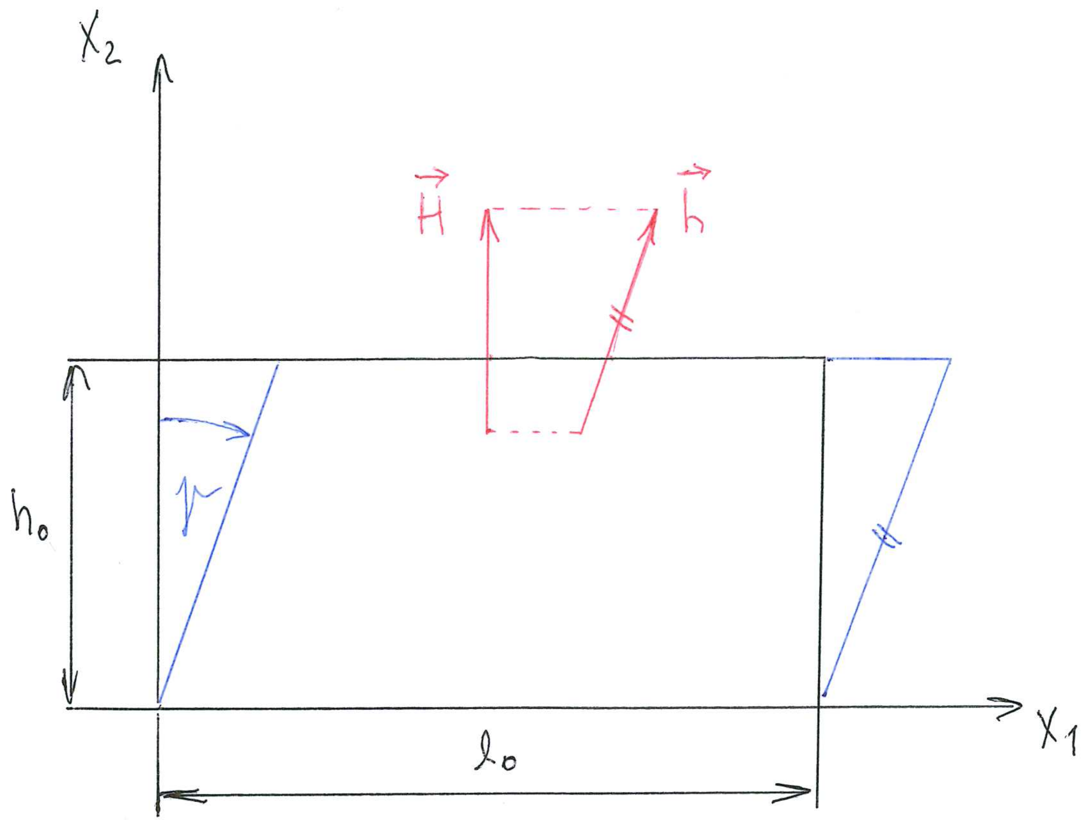
$$= \int_{\Delta S_0} H_jN_j\,\mathrm{d}S_0 = \int_{\Delta S_0} \mathbf{H}\cdot\mathbf{N}\,\mathrm{d}S_0$$

Definition

$$H_j = Jh_iF_{ij}^{-T} = JF_{ji}^{-1}h_i \quad \Rightarrow \quad \boxed{\mathbf{H} = J\mathbf{F}^{-1}\mathbf{h}}$$

Example 3: Simple shear



$$\underline{F} = \begin{bmatrix} 1 & \operatorname{tg}\gamma \\ 0 & 1 \end{bmatrix}$$

choose $\underline{H} = \begin{Bmatrix} 0 \\ \dot{q} \end{Bmatrix}$

$$\underline{h} = \frac{1}{J}\underline{F}\,\underline{H} = \begin{Bmatrix} \dot{q}\operatorname{tg}\gamma \\ \dot{q} \end{Bmatrix}$$

Flow follows material lines. Heat flux is conserved.

2nd Piola-Kirchhoff stress tensor (1/2)

Green-Lagrange strain

$$\dot{\mathrm{e}} = \mathbf{F}^T\mathbf{D}\mathbf{F} \quad \Rightarrow \quad \mathbf{D} = \mathbf{F}^{-T}\dot{\mathrm{e}}\mathbf{F}^{-1}$$

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Index notation

$$D_{ij} = F_{ik}^{-T} \dot{e}_{kl} F_{lj}^{-1} = F_{ki}^{-1} \dot{e}_{kl} F_{jl}^{-T}$$

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Power density in the reference configuration

$J\boldsymbol{\sigma}{:}\mathbf{D} = J\sigma_{ij} D_{ij} = J F_{ki}^{-1} \sigma_{ij} F_{jl}^{-T} \dot{e}_{kl}$

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$$J\boldsymbol{\sigma}:\mathbf{D} = J\sigma_{ij} D_{ij} = \underbrace{J F_{ki}^{-1} \sigma_{ij} F_{jl}^{-T}}_{S_{kl}} \dot{e}_{kl} = \mathbf{S} : \dot{\mathbf{e}}$$

2nd Piola-Kirchhoff stress tensor (1/2)

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Definition

$$\boxed{\mathbf{S} = J \mathbf{F}^{-1} \boldsymbol{\sigma} \mathbf{F}^{-T}} \qquad \text{sym}$$

2nd Piola-Kirchhoff stress tensor (2/2)

Transformation of stress power

$$\boxed{\int_{V_t} \boldsymbol{\sigma} : \mathbf{D} \, \mathrm{d}V_t = \int_{V_0} \mathbf{S} : \dot{\mathbf{e}} \, \mathrm{d}V_0}$$

2nd Piola-Kirchhoff stress tensor (2/2)

Transformation of stress power

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Remarks

- $\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-T} \quad \Rightarrow \quad \mathbf{S} = \mathbf{F}^{-1}\mathbf{P}$ (symmetrization)
- 1st-PK comes from stress equilibrium.
- 2nd-PK comes from stress power.

Conjugate tensors

In general

$$\dot{\mathrm{E}} = \mathcal{L}(\mathrm{D}) \quad \Rightarrow \quad \forall \mathbf{E}, \exists \boldsymbol{\Sigma} : \;\; J\boldsymbol{\sigma} : \mathrm{D} = \boldsymbol{\Sigma} : \dot{\mathrm{E}}$$

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Definition

$$\boxed{\{\mathbf{\Sigma}, \mathbf{E}\} \text{ conjugate} \quad \Longleftrightarrow \quad \int_{V_t} \boldsymbol{\sigma}\!:\!\mathbf{D}\,\mathrm{d}V_t = \int_{V_0} \mathbf{\Sigma}\!:\!\dot{\mathbf{E}}\,\mathrm{d}V_0}$$

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Examples: $\{\mathbf{S}, \mathbf{E}\}$, $\{\mathbf{P}, \mathbf{F}\}$, $\{\boldsymbol{\sigma}, \quad \}$

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Examples: $\{\mathbf{S}, \mathbf{E}\}$, $\{\mathbf{P}, \mathbf{F}\}$, $\{\boldsymbol{\sigma}, -\}$, $\{\boldsymbol{\sigma}, \boldsymbol{\epsilon}\}$ = approximation

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Examples: $\{\mathbf{S}, \mathbf{E}\}$, $\{\mathbf{P}, \mathbf{F}\}$, $\{\boldsymbol{\sigma}, -\}$, $\{\boldsymbol{\sigma}, \boldsymbol{\epsilon}\}$ = approximation

Material equation of heat conduction

$$\boxed{\mathcal{K} - \mathrm{Div}\, \mathbf{H} + \boldsymbol{\Sigma} : \dot{\mathbf{E}} = \rho_0 \dot{u}} \qquad \text{in} \quad \Omega_0$$

5. Clausius-Duhem inequality

Spatial description

Integral formulation

$$\frac{\mathrm{d}}{\mathrm{d}t} S \geq \int_{V_t} \frac{\kappa}{T} \,\mathrm{d}V_t - \int_{S_t} \frac{\mathbf{h} \cdot \mathbf{n}}{T} \,\mathrm{d}S_t$$

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Local form

$$\rho \dot{\eta} \geq \frac{\kappa}{T} - \mathrm{div} \left(\frac{\mathbf{h}}{T} \right)$$

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$$\operatorname{div}\left(\frac{\mathbf{h}}{T}\right) = \frac{\partial}{\partial x_i}\left(\frac{h_i}{T}\right)$$

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$$\operatorname{div}\left(\frac{\mathbf{h}}{T}\right) = \frac{\partial}{\partial x_i}\left(\frac{h_i}{T}\right) = \frac{1}{T^2}\left(\frac{\partial h_i}{\partial x_i}T - h_i\frac{\partial T}{\partial x_i}\right)$$

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Local CD inequality

$$\boxed{\rho \dot{\eta} \geq \frac{\kappa - \operatorname{div} \mathbf{h}}{T} + \frac{1}{T^2} \mathbf{h} \cdot \operatorname{grad} T} \qquad \text{in} \quad \Omega$$

Spatial description

Integral formulation

$$\frac{\mathrm{d}}{\mathrm{d}t} S \geq \int_{V_t} \frac{\kappa}{T} \, \mathrm{d}V_t - \int_{S_t} \frac{\mathbf{h} \cdot \mathbf{n}}{T} \, \mathrm{d}S_t$$

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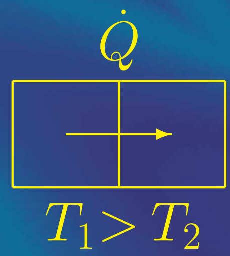
Interpretation of the first r.h.s. term is easy. What about the second one?

Rudolf J. E. Clausius 1822-1888

Why is the sky blue?

Conceptual example



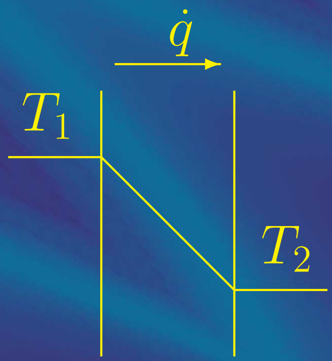
$$\dot{S}_1 = -\frac{\dot{Q}}{T_1}$$

$$\dot{S}_2 = +\frac{\dot{Q}}{T_2}$$

$$\dot{S} = \dot{S}_1 + \dot{S}_2 = \dot{Q}\left(\frac{1}{T_2} - \frac{1}{T_1}\right) > 0$$

- Entropy of a closed system increases.
- Equal sign '$=$' used in this definition.

Example: Heat conduction in a slab



$$\frac{\mathrm{d}}{\mathrm{d}t} S \geq -\int_{S_t} \frac{\mathbf{h}\cdot\mathbf{n}}{T}\,\mathrm{d}S_t = \frac{\dot{q}}{T_1} - \frac{\dot{q}}{T_2}$$

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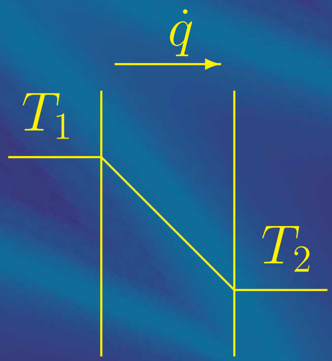

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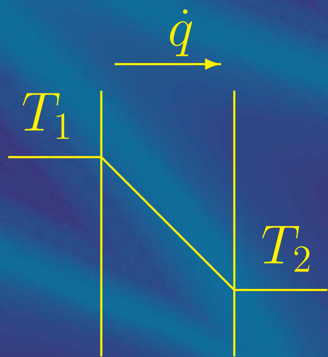

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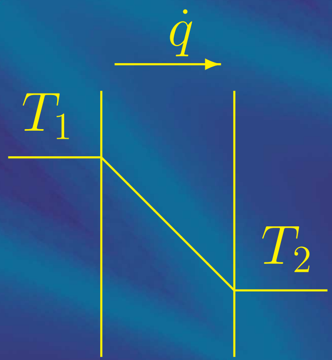
$$\frac{\mathrm{d}}{\mathrm{d}t}S = 0 \geq \dot{q}\left(\frac{1}{T_1} - \frac{1}{T_2}\right)$$

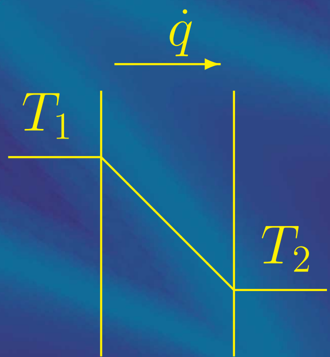
Example: Heat conduction in a slab



$$\frac{\mathrm{d}}{\mathrm{d}t} S \geq - \int_{S_t} \frac{\mathbf{h} \cdot \mathbf{n}}{T} \, \mathrm{d}S_t = \frac{\dot{q}}{T_1} - \frac{\dot{q}}{T_2} < 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t} S = 0 \geq \dot{q} \left(\frac{1}{T_1} - \frac{1}{T_2} \right) \qquad \Rightarrow \qquad \dot{q} \geq 0$$

Example: Heat conduction in a slab



$$\frac{\mathrm{d}}{\mathrm{d}t}S \geq -\int_{S_t} \frac{\mathbf{h}\cdot\mathbf{n}}{T}\,\mathrm{d}S_t = \frac{\dot{q}}{T_1} - \frac{\dot{q}}{T_2} < 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}S = 0 \geq \dot{q}\left(\frac{1}{T_1} - \frac{1}{T_2}\right) \quad \Rightarrow \quad \dot{q} \geq 0$$

- CD inequality actually determines the heat vector direction.
- Admits non-physical solutions (necessary but not sufficient cnd.).

Dissipation inequality

Spatial description

$$\boxed{-\rho\eta\dot{T} + \boldsymbol{\sigma}:\mathbf{D} - \frac{1}{T}\mathbf{h}\cdot\mathrm{grad}\,T \geq \rho\dot{\psi}} \qquad \text{in} \quad \Omega$$

Material description

$$\boxed{-\rho_0\eta\dot{T} + \boldsymbol{\Sigma}:\dot{\mathbf{E}} - \frac{1}{T}\mathbf{H}\cdot\mathrm{Grad}\,T \geq \rho_0\dot{\psi}} \qquad \text{in} \quad \Omega_0$$

$\psi = u - T\eta$ (the Helmholtz free energy)

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Discussion of the second law (optional)

Clausius-Planck inequality

$$\underset{T}{\boxed{\mathrm{d}V_t}} \xleftarrow{\;(\kappa - \mathrm{div}\mathbf{h})\,\mathrm{d}V_t\;}$$

$$\rho\dot{\eta} \geq \frac{\kappa - \mathrm{div}\mathbf{h}}{T}$$

Clausius-Planck inequality

$$\underset{T}{\boxed{\mathrm{d}V_t}} \xleftarrow{\;(\kappa - \mathrm{div}\mathbf{h})\,\mathrm{d}V_t\;} \qquad\qquad \rho\dot{\eta} \geq \frac{\kappa - \mathrm{div}\mathbf{h}}{T}$$

Integration over an arbitrary subdomain

$$\frac{\mathrm{d}}{\mathrm{d}t} S \geq \int_{V_t} \frac{\kappa}{T}\,\mathrm{d}V_t - \int_{S_t} \frac{\mathbf{h}\cdot\mathbf{n}}{T}\,\mathrm{d}S_t - \int_{V_t} \frac{\mathbf{h}\cdot\mathrm{grad}T}{T^2}\,\mathrm{d}V_t$$

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Integration over an arbitrary subdomain

$$\frac{\mathrm{d}}{\mathrm{d}t}S \geq \int_{V_t} \frac{\kappa}{T}\,\mathrm{d}V_t - \int_{S_t} \frac{\mathbf{h}\cdot\mathbf{n}}{T}\,\mathrm{d}S_t - \underbrace{\int_{V_t} \frac{\mathbf{h}\cdot\mathrm{grad}T}{T^2}\,\mathrm{d}V_t}_{\text{additional term}}$$

Note that CPI is naturaly defined locally whereas CDI totally. However, while the local CPI has a clear physical meaning, the CDI's flux integral defies comprehension.

Clausius-Planck inequality

$$\underset{T}{\boxed{\mathrm{d}V_t}} \xleftarrow{\;(\kappa - \mathrm{div}\mathbf{h})\,\mathrm{d}V_t\;}$$

$$\rho\dot{\eta} \geq \frac{\kappa - \mathrm{div}\mathbf{h}}{T}$$

Integration over the whole body

$$\frac{\mathrm{d}}{\mathrm{d}t}S \geq \int_{\Omega_t} \frac{\kappa}{T}\,\mathrm{d}V_t - \int_{\Gamma_t} \frac{\mathbf{h}\cdot\mathbf{n}}{T}\,\mathrm{d}S_t - \int_{\Omega_t} \frac{\mathbf{h}\cdot\mathrm{grad}T}{T^2}\,\mathrm{d}V_t$$

Clausius-Planck inequality

$$\underset{T}{\boxed{\mathrm{d}V_t}} \xleftarrow{(\kappa - \mathrm{div}\,\mathbf{h})\,\mathrm{d}V_t} \qquad\qquad \rho\dot{\eta} \geq \frac{\kappa - \mathrm{div}\,\mathbf{h}}{T}$$

Integration over a closed system

$$\frac{\mathrm{d}}{\mathrm{d}t} S \geq -\int_{\Omega_t} \frac{\mathbf{h}\cdot\mathrm{grad}\,T}{T^2}\,\mathrm{d}V_t$$

Clausius-Planck inequality

$$\underset{T}{\boxed{\mathrm{d}V_t}} \xleftarrow{(\kappa - \mathrm{div}\mathbf{h})\,\mathrm{d}V_t} \qquad\qquad \rho\dot{\eta} \geq \frac{\kappa - \mathrm{div}\mathbf{h}}{T}$$

Integration over a closed system

$$\frac{\mathrm{d}}{\mathrm{d}t} S \geq -\int_{\Omega_t} \frac{\mathbf{h}\cdot\mathrm{grad}T}{T^2}\,\mathrm{d}V_t$$

Sufficient condition

$$\boxed{\mathbf{h}\cdot\mathrm{grad}T \leq 0}$$ (Fourier inequality)

Discussion (Truesdell)

$\boxed{\text{Clausius-Planck}}$

$$\rho\dot{\eta} \geq \frac{\kappa - \mathrm{div}\mathbf{h}}{T}$$

$\boxed{\text{Fourier}}$

$$\mathbf{h}\cdot\mathrm{grad}T \leq 0$$

Discussion (Truesdell)

Clausius-Planck

$$\rho\dot{\eta} \geq \frac{\kappa - \mathrm{div}\mathbf{h}}{T}$$

Fourier

$$\frac{\mathbf{h}\cdot\mathrm{grad}T}{T^2} \leq 0$$

Discussion (Truesdell)

$\boxed{\text{Clausius-Planck}}$

$$\rho\dot{\eta} \geq \frac{\kappa - \mathrm{div}\mathbf{h}}{T}$$

$\boxed{\text{Fourier}}$

$$\frac{\mathbf{h}\cdot\mathrm{grad}T}{T^2} \leq 0$$

$\boxed{\text{Clausius-Duhem}}$

$$\rho\dot{\eta} \geq \frac{\kappa - \mathrm{div}\mathbf{h}}{T} + \frac{\mathbf{h}\cdot\mathrm{grad}T}{T^2}$$

Recap

- Clausius-Duhem inequality (CDI) does not have direct physical intepretation.
- CDI pitfalls: meaning of the entropy flux, integration over arbitrary volume.
- CDI determines the heat vector direction.
- CDI admits non-physical solutions.
- Employing Clausius-Planck inequality (CPI) is an option.
- If a model is independent of ∇T then CDI reduces to CPI.
- CPI $+$ local Fourier inequality implies CDI.