EULER DESCRIPTION II

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Contents

- Material derivative
- Velocity gradient
    - Additive decomposition
    - Rate of deformation
    - Vorticity (spin)

Material derivative (1/4)

Given a general field $\mathbf{T}$ = scalar, vector, tensor, ...

$$\mathbf{T} = \phi(\mathbf{X}, t) = \varphi(\mathbf{x}, t)$$

subject to $\mathbf{X} \leftrightarrow \mathbf{x}$.

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$$\phi(\mathbf{X}, t) \equiv \varphi(\mathbf{x}(\mathbf{X}, t), t), \qquad (\text{that is } \phi = \varphi \circ \mathbf{x})$$

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Definition

$$\boxed{\dot{\mathbf{T}} = \frac{\partial}{\partial t} \phi(\mathbf{X}, t)}$$

Material derivative (2/4)

Identity

$$\phi(\mathbf{X},t) \equiv \varphi(\mathbf{x}(\mathbf{X},t),t)$$

Chain rule calculation

$$\frac{\partial}{\partial t}\phi(\mathbf{X},t) \equiv \frac{\partial}{\partial t}\varphi(\mathbf{x}(\mathbf{X},t),t) = \frac{\partial \varphi}{\partial x_j}\frac{\partial x_j}{\partial t} + \frac{\partial \varphi}{\partial t}$$

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Euler representation

$$\boxed{\dot{\mathbf{T}} = \frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial x_j}v_j}$$

Material derivative (3/4)

Material derivative: $\dfrac{\partial \phi}{\partial t} = \left(\dfrac{\partial \mathbf{T}}{\partial t} \right)_X = \dot{\mathbf{T}}$

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Convective derivative: $\dfrac{\partial \varphi}{\partial x_j} v_j = (\mathrm{grad}\,\mathbf{T})\mathbf{v}, \quad \mathrm{grad}\,\mathbf{T} : R^3 \to R^N$

. . . where N is the order of $\mathbf{T}$.

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Euler (spatial) representation

$$\dot{\mathbf{T}} = \left(\frac{\partial \mathbf{T}}{\partial t}\right)_x + (\operatorname{grad} \mathbf{T})\mathbf{v}$$

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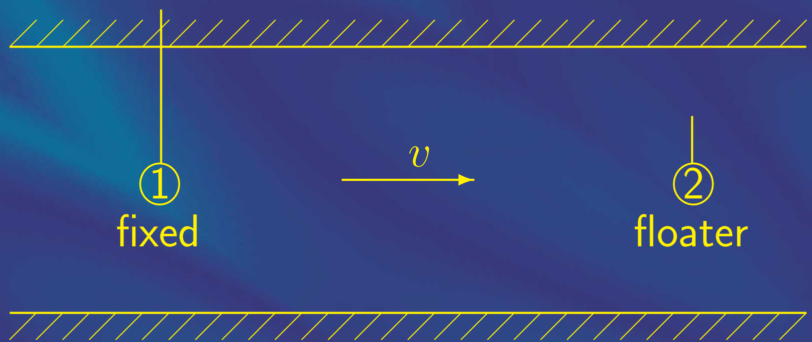
$$\dot{\mathbf{T}} = \left(\frac{\partial \mathbf{T}}{\partial t}\right)_x + (\operatorname{grad} \mathbf{T})\mathbf{v}$$

Notation: $\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{u}}$

Material derivative (4/4)

Flow in a channel with two thermometers



$$\frac{\mathrm{d}T_1}{\mathrm{d}t} \qquad \qquad \frac{\mathrm{d}T_2}{\mathrm{d}t}$$

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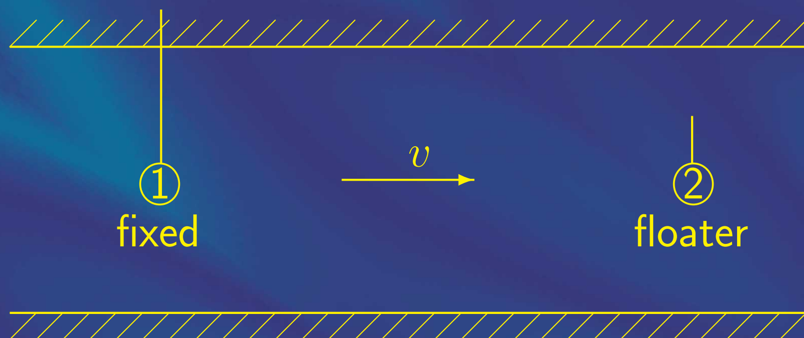

$$\frac{\mathrm{d}T_1}{\mathrm{d}t} \quad \text{(local)}$$

$$\frac{\mathrm{d}T_2}{\mathrm{d}t} = \dot{T}_2 \quad \text{(material)}$$

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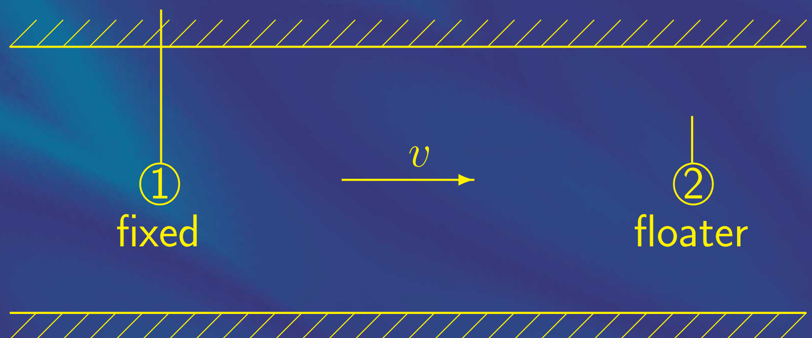

$$\frac{\mathrm{d}T_1}{\mathrm{d}t} \quad \text{(local)} \qquad\qquad \frac{\mathrm{d}T_2}{\mathrm{d}t} = \dot{T}_2 \quad \text{(material)}$$

Isochoric process: $\dot{q} = c\dot{T}$ (to be proved later); insulation $\Rightarrow \dot{T} = 0$.

Velocity gradient (1/2)

Component definition

$$L_{ij} = \frac{\partial v_i}{\partial x_j} \;, \quad \mathbf{L} = \operatorname{grad} \mathbf{v}$$

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Important relation

$$\boxed{[L] = [\dot{F}][F]^{-1}}$$

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Proof

$$L_{ij} = \frac{\partial v_i}{\partial x_j} = \frac{\partial v_i}{\partial X_k}\frac{\partial X_k}{\partial x_j} = \frac{\partial \dot{x}_i}{\partial X_k}F^{-1}_{kj} = \dot{F}_{ik}F^{-1}_{kj}$$

Velocity gradient (2/2)

Differential

$$\boxed{\mathrm{d}\{v\} = [L]\,\mathrm{d}\{x\}}$$

Additive decomposition

$$[L] = \frac{1}{2}(\,[L] + [L]^T) + \frac{1}{2}(\,[L] - [L]^T)$$

Velocity gradient (2/2)

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$$\boxed{\mathrm{d}\{v\} = [L]\,\mathrm{d}\{x\}}$$

Additive decomposition

$$[L] = \underbrace{\frac{1}{2}\big([L] + [L]^T\big)}_{[D]\ \text{sym}} + \underbrace{\frac{1}{2}\big([L] - [L]^T\big)}_{[W]\ \text{skew sym}}$$

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$$[W]\{x\} \perp \{x\} \quad \text{for} \quad \forall\{x\}$$

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Proof: $\{x\}^T[W]\{x\} = ([W]\{x\})^T\{x\} = \{x\}^T[W]^T\{x\} = -\{x\}^T[W]\{x\}$

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Rate of deformation (1/5)

Material line segment:

choose fixed $\mathbf{X}_0\,,\ \ \mathrm{d}\mathbf{X}\ \mapsto\ \mathrm{d}\mathbf{x}(t)$

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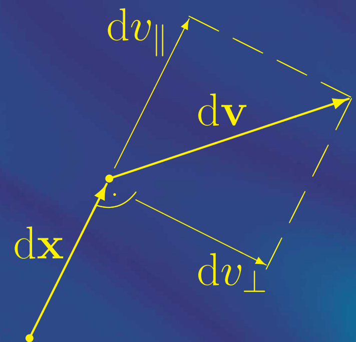

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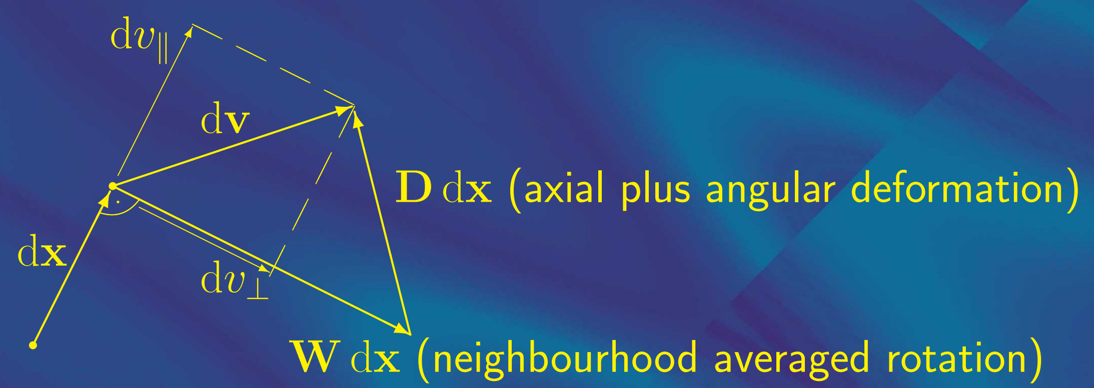

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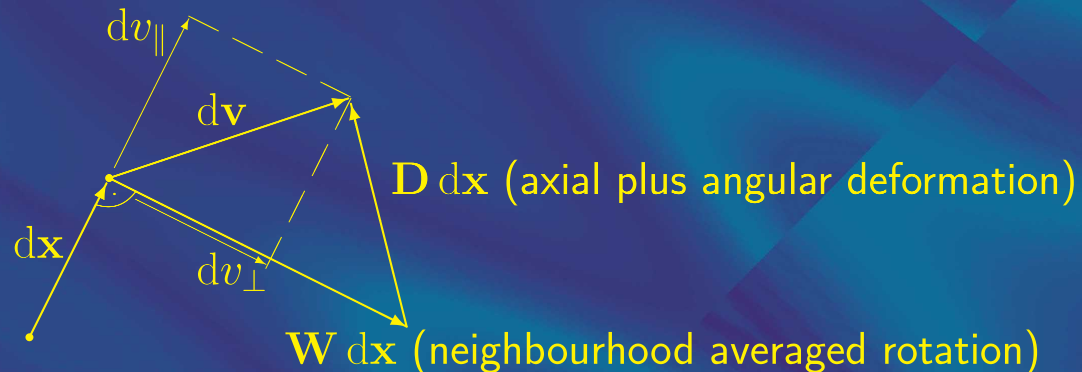

Denote $\mathrm{d}l = \|\,\mathrm{d}\mathbf{x}\| = \text{function}(t)$

Rate of deformation (2/5)

For a fixed material point

$$\frac{\mathrm{d}}{\mathrm{d}t}(\,\mathrm{d}l)^2 = \frac{\mathrm{d}}{\mathrm{d}t}(\,\mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x}) = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\cdot\mathrm{d}\mathbf{x} + \,\mathrm{d}\mathbf{x}\cdot\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = 2\,\mathrm{d}\mathbf{x}\cdot\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = 2\,\mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{v}$$

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Quadratic form

$\frac{\mathrm{d}}{\mathrm{d}t}(\,\mathrm{d}l)^2 = 2\,\mathrm{d}\{x\}^T\,\mathrm{d}\{v\}$

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Quadratic form

$$\frac{\mathrm{d}}{\mathrm{d}t}(\,\mathrm{d}l)^2 = 2\,\mathrm{d}\{x\}^T\,\mathrm{d}\{v\} = 2\,\mathrm{d}\{x\}^T[L]\,\mathrm{d}\{x\} = 2\,\mathrm{d}\{x\}^T(\,[D]+[W]\,)\,\mathrm{d}\{x\}$$

Rate of deformation (2/5)

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Theorem

$$\boxed{\mathbf{D} = \mathbf{0} \quad\Longleftrightarrow\quad \forall\,\mathrm{d}\mathbf{x} :\ \mathrm{d}l = \text{const.}}$$

Rate of deformation (3/5)

Index notation

$$D_{ij} = \frac{1}{2}(L_{ij} + L_{ji})$$

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$$D_{ij} = \frac{1}{2}(L_{ij} + L_{ji}) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

Rate of deformation (3/5)

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$\int_0^t \mathbf{D}\,\mathrm{d}\tau$ depends on integration path

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$\int_0^t \mathbf{D}\,\mathrm{d}\tau$ depends on integration path

Corollary: $\dot{\mathbf{E}} \neq \mathbf{D} \neq \dot{\mathbf{A}}$ (the rate of deformation is not a strain rate)

Rate of deformation (4/5)

Green-Lagrange strain

$$[\dot{e}] = \frac{1}{2}([F]^T[F] - [I])^{\cdot} = \frac{1}{2}([\dot{F}]^T[F] + [F]^T[\dot{F}])$$

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Hence

$$[F]^{-T}[\dot{e}][F]^{-1} = \frac{1}{2}([F]^{-T}[\dot{F}]^T + [\dot{F}][F]^{-1})$$

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$$\boxed{[\dot{e}] = [F]^T[D][F]}$$

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$$\boxed{[\dot{e}] = [F]^T[D][F]}$$

In general : $\dot{\mathbf{E}} = \mathcal{L}(\mathbf{D}) \quad \Rightarrow \quad \mathbf{D} = \mathbf{0} \Leftrightarrow \dot{\mathbf{E}} = \mathbf{0}$

Rate of deformation (4/5)

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Remark: $\mathbf{E}$ is a two-point tensor, therefore $\dot{\mathbf{E}}$ depends both on Ω_t and Ω_0.

Rate of deformation (5/5)

Initial conditions

$$\Omega_0 : [F] = [I] \ , \quad [\dot{F}] \neq [0]$$

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Therefore

$$[L] = [\dot{F}][F]^{-1} = [\dot{F}] = [\dot{z}]$$

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Therefore

$$[L] = [\dot{F}][F]^{-1} = [\dot{F}] = [\dot{z}]$$

Symmetric part

$$[D] = \frac{1}{2}([L] + [L]^T) = \frac{1}{2}([\dot{z}] + [\dot{z}]^T) = \frac{1}{2}([z] + [z]^T)^{\cdot} = [\dot{\epsilon}]$$

Rate of deformation (5/5)

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$$\Omega_0 : [F] = [I] \ , \quad [\dot{F}] \neq [0]$$

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$$[D] = \frac{1}{2}([L] + [L]^T) = \frac{1}{2}([\dot{z}] + [\dot{z}]^T) = \frac{1}{2}([z] + [z]^T)^{\cdot} = [\dot{\epsilon}]$$

Corollary

At Ω_0 it holds $\dot{\boldsymbol{\epsilon}} = \mathbf{D}$ exactly.

Vorticity (spin) tensor (1/3)

Circular motion

$$\mathrm{d}\mathbf{v} = \boldsymbol{\omega} \times \mathrm{d}\mathbf{x}$$

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Tullio Levi-Civita (1873-1941)

$$\mathrm{d}v_i = \gamma_{ijk}\,\omega_j\,\mathrm{d}x_k$$

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Component definition

$$W_{ik} = \gamma_{ijk}\,\omega_j \qquad \text{(skew sym)}$$

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$$\mathrm{d}\mathbf{v} = \boldsymbol{\omega} \times \mathrm{d}\mathbf{x}$$

Tullio Levi-Civita (1873-1941)

$$\mathrm{d}v_i = \underbrace{\gamma_{ijk}\,\omega_j}_{W_{ik}}\,\mathrm{d}x_k = W_{ik}\,\mathrm{d}x_k$$

Component definition

$$W_{ik} = \gamma_{ijk}\,\omega_j \qquad \text{(skew sym)}$$

Algebraic expression for the cross product

$$\boxed{\{v\} = [W]\,\mathrm{d}\{x\}}$$

Vorticity (spin) tensor (2/3)

Matrix notation

$$[W] = \begin{bmatrix} 0 & \gamma_{132}\omega_3 & \gamma_{123}\omega_2 \\ & 0 & \\ & & 0 \end{bmatrix}$$

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Matrix notation

$$[W] = \begin{bmatrix} 0 & \gamma_{132}\omega_3 & \gamma_{123}\omega_2 \\ \gamma_{231}\omega_3 & 0 & \gamma_{213}\omega_1 \\ \gamma_{321}\omega_2 & \gamma_{312}\omega_1 & 0 \end{bmatrix}$$

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Skew symmetry

$$[W] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ & 0 & -\omega_1 \\ & & 0 \end{bmatrix}$$

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Axial vector

$$\{\omega\} = \begin{Bmatrix} -W_{23} \\ W_{13} \\ -W_{12} \end{Bmatrix}$$

Vorticity (spin) tensor (2/3)

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Axial vector

$$\{\omega\} = \begin{Bmatrix} -W_{23} \\ W_{13} \\ -W_{12} \end{Bmatrix}$$

... transforms as a vector for SO$(3)^+$ rotation group.

Vorticity (spin) tensor (3/3)

Circulation integral

$$\oint \mathbf{v} \cdot \mathrm{d}\mathbf{l} = \int_{\Delta S_t} \mathsf{curl}\, \mathbf{v} \, \mathrm{d}S_t$$

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$$\oint \mathbf{v} \cdot \mathrm{d}\mathbf{l} = \int_{\Delta S_t} \mathsf{curl}\, \mathbf{v}\, \mathrm{d}S_t$$

Local mean circulation

$$\lim_{\Delta S_t \to 0} \frac{1}{\Delta S_t} \int_{\Delta S_t} \mathsf{curl}\, \mathbf{v}\, \mathrm{d}S_t = \mathsf{curl}\, \mathbf{v} \qquad \text{(equivalent to } 2\mathbf{W}\text{)}$$

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Circular motion

$$\boldsymbol{\omega} = \frac{\mathbf{v}}{r} = \frac{2\pi r \mathbf{v}}{2\pi r^2}$$

Vorticity (spin) tensor (3/3)

Circulation integral

$$\oint \mathbf{v}\cdot \mathrm{d}\mathbf{l} = \int_{\Delta S_t} \mathsf{curl}\, \mathbf{v}\, \mathrm{d}S_t$$

Local mean circulation

$$\lim_{\Delta S_t \to 0} \frac{1}{\Delta S_t} \int_{\Delta S_t} \mathsf{curl}\, \mathbf{v}\, \mathrm{d}S_t = \mathsf{curl}\, \mathbf{v} \qquad \text{(equivalent to } 2\mathbf{W}\text{)}$$

Circular motion

$$\boldsymbol{\omega} = \frac{\mathbf{v}}{r} = \frac{2\pi r \mathbf{v}}{2\pi r^2} = \frac{1}{2\Delta S_t} \oint \mathbf{v}\cdot \mathrm{d}\mathbf{l}$$

Vorticity (spin) tensor (3/3)

Circulation integral

$$\oint \mathbf{v} \cdot \mathrm{d}\mathbf{l} = \int_{\Delta S_t} \mathsf{curl}\, \mathbf{v}\, \mathrm{d}S_t$$

Local mean circulation

$$\lim_{\Delta S_t \to 0} \frac{1}{\Delta S_t} \int_{\Delta S_t} \mathsf{curl}\, \mathbf{v}\, \mathrm{d}S_t = \mathsf{curl}\, \mathbf{v} \qquad (\text{equivalent to } 2\mathbf{W})$$

Circular motion

$$\boldsymbol{\omega} = \frac{\mathbf{v}}{r} = \frac{2\pi r \mathbf{v}}{2\pi r^2} = \frac{1}{2\Delta S_t} \oint \mathbf{v} \cdot \mathrm{d}\mathbf{l} \to \frac{1}{2} \mathsf{curl}\, \mathbf{v} \qquad (\text{equivalent to } \mathbf{W})$$

Vorticity (spin) tensor (3/3)

Circulation integral

$$\oint \mathbf{v} \cdot \mathrm{d}\mathbf{l} = \int_{\Delta S_t} \mathsf{curl}\,\mathbf{v}\,\mathrm{d}S_t$$

Local mean circulation

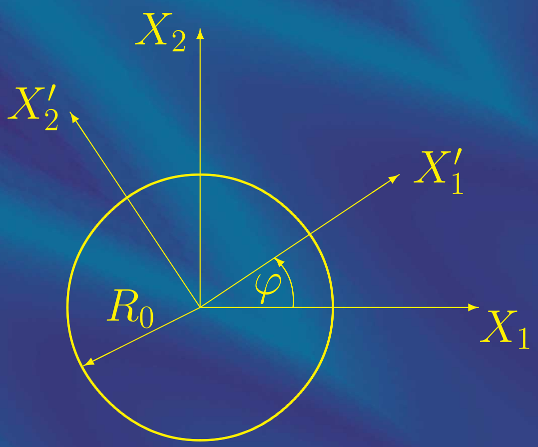
$$\lim_{\Delta S_t \to 0} \frac{1}{\Delta S_t} \int_{\Delta S_t} \mathsf{curl}\,\mathbf{v}\,\mathrm{d}S_t = \mathsf{curl}\,\mathbf{v} \qquad (\text{equivalent to } 2\mathbf{W})$$

Circular motion

$$\boldsymbol{\omega} = \frac{\mathbf{v}}{r} = \frac{2\pi r \mathbf{v}}{2\pi r^2} = \frac{1}{2\Delta S_t} \oint \mathbf{v} \cdot \mathrm{d}\mathbf{l} \to \frac{1}{2}\mathsf{curl}\,\mathbf{v} \qquad (\text{equivalent to } \mathbf{W})$$

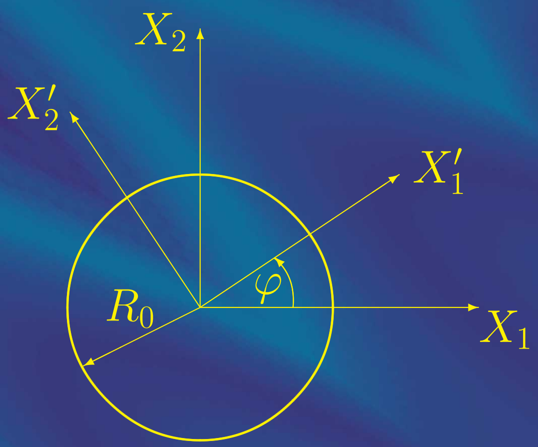
$$\boxed{\mathrm{d}\mathbf{v} = \mathbf{W}\,\mathrm{d}\mathbf{x} = \boldsymbol{\omega} \times \mathrm{d}\mathbf{x}}$$

Example 4: Rotation



$$u_1(X_1, X_2) = X_1(\cos\varphi - 1) - X_2 \sin\varphi$$
$$u_2(X_1, X_2) = X_1 \sin\varphi + X_2(\cos\varphi - 1)$$

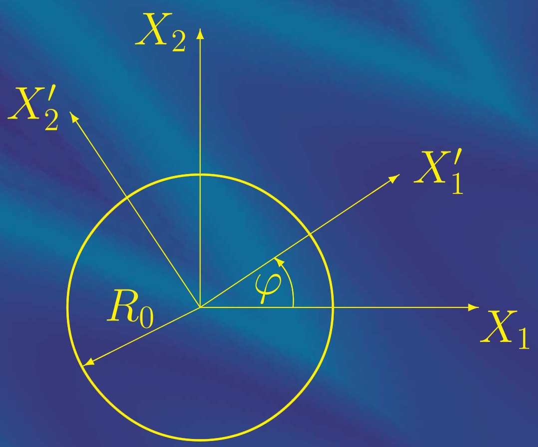
Example 4: Rotation



$$u_1(X_1, X_2) = X_1(\cos\omega t - 1) - X_2 \sin\omega t$$

$$u_2(X_1, X_2) = X_1 \sin\omega t + X_2(\cos\omega t - 1)$$

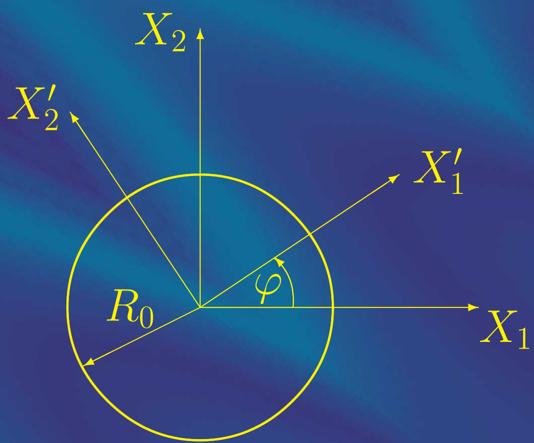
Example 4: Rotation



$$u_1 = X_1(\cos\omega t - 1) - X_2 \sin\omega t$$

$$u_2 = X_1 \sin\omega t + X_2(\cos\omega t - 1)$$

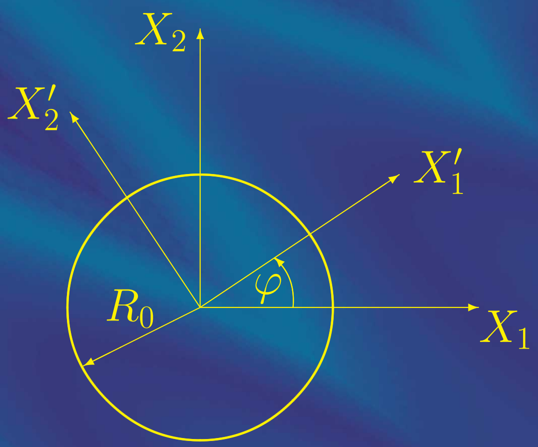
Example 4: Rotation



$$u_1 + X_1 = X_1 \cos \omega t - X_2 \sin \omega t$$

$$u_2 + X_2 = X_1 \sin \omega t + X_2 \cos \omega t$$

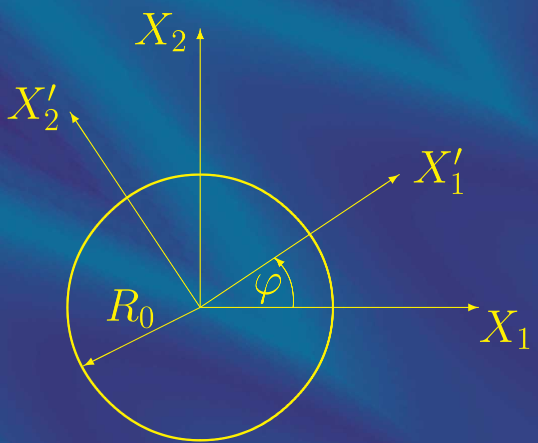
Example 4: Rotation



$$x_1 = X_1 \cos \omega t - X_2 \sin \omega t$$

$$x_2 = X_1 \sin \omega t + X_2 \cos \omega t$$

Example 4: Rotation



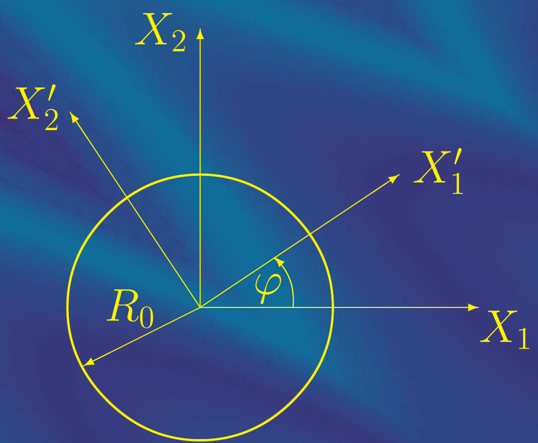
$$x_1 = X_1 \cos \omega t - X_2 \sin \omega t$$

$$x_2 = X_1 \sin \omega t + X_2 \cos \omega t$$

$$v_1 = -\omega X_1 \sin \omega t - \omega X_2 \cos \omega t$$

$$v_2 = \omega X_1 \cos \omega t - \omega X_2 \sin \omega$$

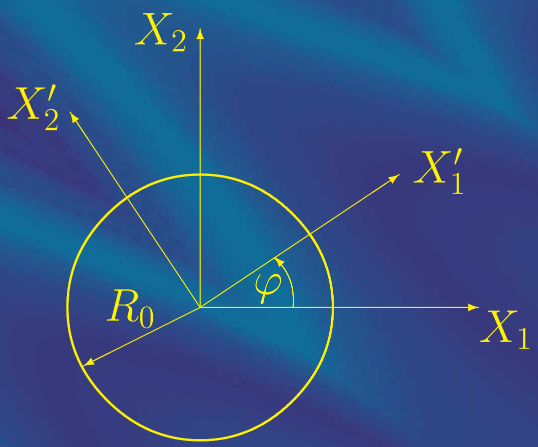
Example 4: Rotation



$$x_1 = X_1 \cos \omega t - X_2 \sin \omega t$$
$$x_2 = X_1 \sin \omega t + X_2 \cos \omega t$$

$$v_1 = -\omega(X_1 \sin \omega t + X_2 \cos \omega t)$$
$$v_2 = \omega(X_1 \cos \omega t - X_2 \sin \omega)$$

Example 4: Rotation



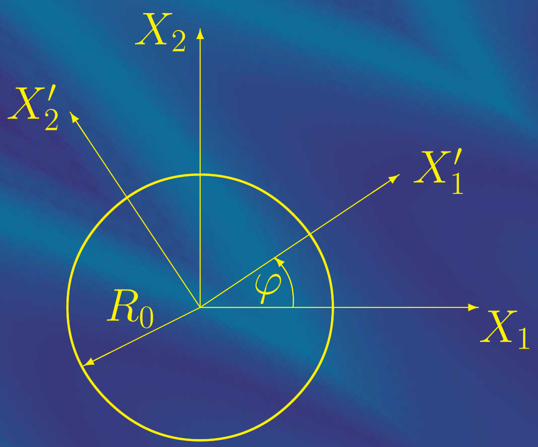
$$x_1 = X_1 \cos\omega t - X_2 \sin\omega t$$

$$x_2 = X_1 \sin\omega t + X_2 \cos\omega t$$

$$v_1 = -\omega(X_1 \sin\omega t + X_2 \cos\omega t) = -\omega x_2$$

$$v_2 = \omega(X_1 \cos\omega t - X_2 \sin\omega) = \omega x_1$$

Example 4: Rotation



$$x_1 = X_1 \cos \omega t - X_2 \sin \omega t$$

$$x_2 = X_1 \sin \omega t + X_2 \cos \omega t$$

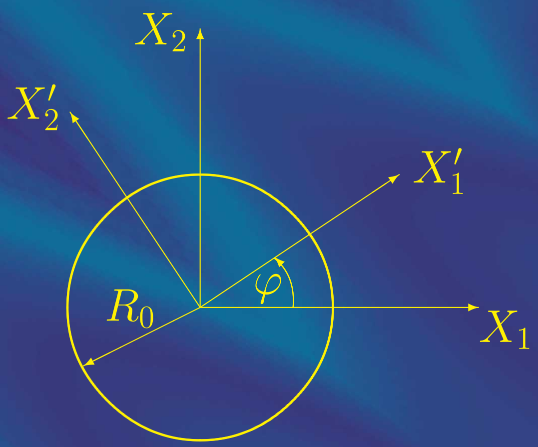
$$v_1 = -\omega (X_1 \sin \omega t + X_2 \cos \omega t) = -\omega x_2$$

$$v_2 = \omega (X_1 \cos \omega t - X_2 \sin \omega) = \omega x_1$$

Acceleration

$$a_1 = \frac{\partial v_1}{\partial t} + \frac{\partial v_1}{\partial x_j} v_j$$

Example 4: Rotation



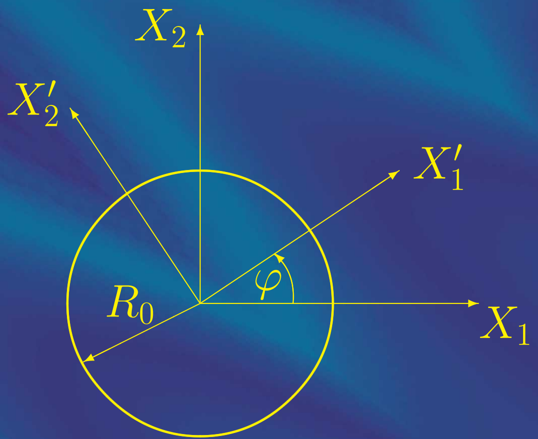
$$x_1 = X_1 \cos \omega t - X_2 \sin \omega t$$
$$x_2 = X_1 \sin \omega t + X_2 \cos \omega t$$

$$v_1 = -\omega(X_1 \sin \omega t + X_2 \cos \omega t) = -\omega x_2$$
$$v_2 = \omega(X_1 \cos \omega t - X_2 \sin \omega) = \omega x_1$$

Acceleration

$$a_1 = \frac{\partial v_1}{\partial t} + \frac{\partial v_1}{\partial x_j} v_j = \frac{\partial v_1}{\partial x_2} v_2$$

Example 4: Rotation



$$x_1 = X_1 \cos \omega t - X_2 \sin \omega t$$

$$x_2 = X_1 \sin \omega t + X_2 \cos \omega t$$

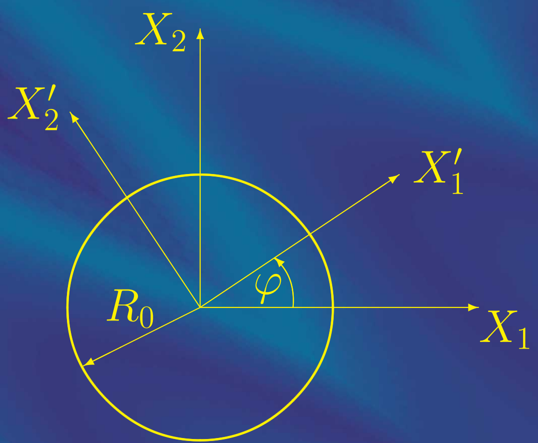
$$v_1 = -\omega(X_1 \sin \omega t + X_2 \cos \omega t) = -\omega x_2$$

$$v_2 = \omega(X_1 \cos \omega t - X_2 \sin \omega) = \omega x_1$$

Acceleration

$$a_1 = \frac{\partial v_1}{\partial t} + \frac{\partial v_1}{\partial x_j} v_j = \frac{\partial v_1}{\partial x_2} v_2 = -\omega(\omega x_1) = -\omega^2 x_1$$

Example 4: Rotation



$$x_1 = X_1 \cos \omega t - X_2 \sin \omega t$$
$$x_2 = X_1 \sin \omega t + X_2 \cos \omega t$$

$$v_1 = -\omega(X_1 \sin \omega t + X_2 \cos \omega t) = -\omega x_2$$
$$v_2 = \omega(X_1 \cos \omega t - X_2 \sin \omega) = \omega x_1$$

Acceleration

$$a_1 = \frac{\partial v_1}{\partial t} + \frac{\partial v_1}{\partial x_j} v_j = \frac{\partial v_1}{\partial x_2} v_2 = -\omega(\omega x_1) = -\omega^2 x_1$$
$$a_2 = \frac{\partial v_2}{\partial t} + \frac{\partial v_2}{\partial x_j} v_j = \frac{\partial v_2}{\partial x_1} x_1 = \omega(-\omega x_2) = -\omega^2 x_2$$

Conclusion: $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{x}$ and $\mathbf{a} = -\omega^2 \mathbf{x}$ (solid mechanics included)