ON NUMERICAL SIMULATION OF FLOW PROBLEMS BY DISCONTINUOUS GALERKIN AND FINITE VOLUME TECHNIQUES

O. Winter, P. Sváček

Department of Technical Mathematics, Faculty of Mechanical Engineering, Center of Advanced Aerospace Technology, Czech Technical University in Prague, Karlovo náměstí 13, 121 35 Praha 2, Česká republika

Abstract

The subject of this contribution is the numerical approximation of the flow problems with aid of OpenFOAM’s implementation of finite volume method and in-house implementation of discontinuous Galerkin method in Julia programming language. The advantage of high-order accurate discontinuous Galerkin method is shown on a case with smooth data. Furthermore, the transonic case is computed with both finite volume and discontinuous Galerkin methods and the results are compared.

Keywords: Euler’s equations, finite volume method, discontinuous Galerkin method, Ringleb’s flow, GAMM channel

1 Introduction

Recently, the finite volume method (FVM) is common approach in the technical practice for the numerical simulations of the flow problems. The FVM is based on the piecewise constant approximation of a solution in each control volume. This method has many strong sides e.g. capability of solving problems on complex geometries, possibility of explicit semi-discrete form and easily achievable conservativity of the method, see e.g. [6]. On the other hand FVM has also some weak sides as e.g. large sensitivity to the computational grid. The fundamental problem of FVM lies in the extension to higher-order accuracy, especially in case of the arbitrary grids, see e.g. [6, 9]. Furthermore, the theoretical analysis of higher-order finite volume method is not developed.

Natural solution to the accuracy problem is the finite element method (FEM) which is based on globally connected local polynomial approximations on each element. However a global statement introduced by FEM formulation may lead to the problems in gas dynamics, i.e., shock waves, see e.g. [9, 16, 5]. One of the method which utilize advantages of both the FVM and FEM is the discontinuous Galerkin method (DGM), i.e., a method which utilize a space of basis and test functions like in the FEM but satisfying the equations in a sense closer to the FVM (jumps at the boundary of the elements).

The DGM can be characterized as a method based on the idea of approximating the solution of a given problem by a piecewise polynomial function over mesh without any requirement of inter-element continuity [4]. This method has some of the characteristics of both methods, e.g. higher modes of polynomial representation need to be filtered in case of appearing of strong gradients (see e.g. [9]) and elliptic terms needs special attention (see e.g. [4, 3]). On the other hand a local formulation permits explicit semi-discrete form, the method provides conservativity of the scheme and allows local high-order polynomial approximation. Furthermore, the theoretical analysis of DGM is possible, see e.g. [3]. For an overview of DGM see e.g. [4] and references inside.

This paper deals with a comparison of the discontinuous Galerkin and finite volume method used for the computation of a selected cases of the inviscid gas dynamics. The remainder of the paper is organized as follows: Section 2 outlines the Euler’s equations of gas dynamics, Section 3 describes the numerical model, i.e., the discontinuous Galerkin method and finite volume method used in this study are briefly introduced. Followed by the section which focuses on the results for chosen test cases, i.e., Ringleb’s flow and GAMM channel. The paper concludes with a summary.
2 Mathematical Model

The inviscid gas dynamics in computational domain $\Omega \subset \mathbb{R}^2$ for any $t \in (0, T)$, $T > 0$ is described by the Euler’s equations, see e.g. [5]. The Euler’s equations written in the conservative form reads

$$
\begin{align*}
\frac{\partial q}{\partial t} + \frac{\partial}{\partial x_j} (gv_j) &= 0, \\
\frac{\partial (gv_j)}{\partial t} + \frac{\partial}{\partial x_j} (gv_jv_j + p\delta_{ij}) &= 0, \quad i = 1, 2 \\
\frac{\partial (\rho e)}{\partial t} + \frac{\partial}{\partial x_j} [(\rho e + p)v_j] &= 0,
\end{align*}
$$

(1)

where the conservative variables are the density $\rho$, the momentum components $\rho v_i$, and the total energy $\rho e$. The system of equations (1) contains also one more variable, the pressure $p$, which is coupled with the system (1) using a constitutive equation. The fluid is assumed to be ideal gas, i.e., the internal energy and the pressure are related through an equation of state $p = p(\rho, \varepsilon)$, which relates the pressure $p$ to the density $\rho$ and to the internal energy $\varepsilon$. The total energy of the gas is the sum of the internal energy $\varepsilon$ and the kinetic energy, i.e.

$$
\varepsilon = \varepsilon + \frac{|v|^2}{2}.
$$

(2)

The internal energy of the ideal gas is related to the thermodynamic temperature $\vartheta$ as

$$
\varepsilon = c_v \vartheta,
$$

(3)

where $c_v$ is the specific heat at a constant volume, see e.g. [8]. The equation of the state of the ideal gas in terms of the density and the internal energy reads

$$
p = (\gamma - 1)\varepsilon,
$$

(4)

where $\gamma$ is the adiabatic index of the gas. The local speed of the sound is then defined as

$$
c = \sqrt{\frac{\gamma p}{\rho}}.
$$

(5)

The Euler’s equations can be written in the vector form

$$
\frac{\partial q}{\partial t} + \frac{\partial F(q)}{\partial x_1} + \frac{\partial G(q)}{\partial x_2} = 0, \quad \text{in } \Omega \times (0, T),
$$

(6)

where $q = [\rho, \rho v_1, \rho v_2, \rho e]^T$ is the vector of the conservative variables (also the state vector) and $F(q), G(q)$ are the non-linear inviscid fluxes given by

$$
F(q) = \begin{bmatrix} \rho v_1 \\ \rho v_1^2 + p \\ \rho v_1 v_2 \\ (\rho e + p)v_1 \end{bmatrix}, \quad G(q) = \begin{bmatrix} \rho v_2 \\ \rho v_1 v_2 \\ \rho v_2^2 + p \\ (\rho e + p)v_2 \end{bmatrix}.
$$

(7)

The system (6) is hyperbolic. It is equipped with the initial condition

$$
q(x, 0) = q^0(x), \quad x \in \Omega,
$$

(8)

and boundary conditions chosen in such a way that problem (6) is well-posed (see, e.g. [5, 2]). To this end, the boundary $\partial \Omega$ is split into three disjoint parts $\Gamma_I, \Gamma_O, \Gamma_W$, representing the inlet, outlet and impermeable/solid walls, respectively.

Inlet $\Gamma_I$ and outlet $\Gamma_O$ boundary conditions are determined according to a regime of flow, i.e., subsonic/supersonic (see, e.g. [5, 2]). We assume three types of boundary conditions:

$^1$Ideal gas with this definition is in some literature referred as calorically perfect gas, see [14].
1. Subsonic inlet: the direction of the velocity (given by the inlet angle), the value of the stagnation density \( \varrho_0 \) and the stagnation pressure \( p_0 \) are prescribed.

Necessary quantities for evaluation of the speed of sound \( c \) and velocity magnitude \( |\mathbf{v}| \) are extrapolated. Using \( |\mathbf{v}| \) and \( c \) the static pressure \( p_T \) and static density \( \varrho_T \) and other variables are computed using the relation between the stagnation and the static quantities, i.e.

\[
p_0 = p_T \left( 1 + \frac{\gamma - 1}{2} \frac{|\mathbf{v}|^2}{c^2} \right)^{\frac{\gamma}{\gamma - 1}} \quad \text{and} \quad \varrho_0 = \varrho_T \left( 1 + \frac{\gamma - 1}{2} \frac{|\mathbf{v}|^2}{c^2} \right)^{\frac{1}{\gamma - 1}}.
\] (9)

2. Subsonic outlet: the conservative variable \( \varrho e \) is prescribed using (2) and (4) through given pressure \( p_O \). Other quantities are extrapolated.

3. Solid wall: reflective boundary condition for all quantities, see [9].

### 3 Numerical Methods

#### 3.1 Discontinuous Galerkin Method

For the discretization with the aid of discontinuous Galerkin method, the computational domain \( \Omega \) is approximated by a polygonal domain \( \Omega_h \). The domain \( \Omega_h \) is then discretized using a triangulation \( \mathcal{T}_h \), which consists of \( K \) closed elements (triangles) \( D_k \), i.e.,

\[
\Omega \approx \Omega_h = \bigcup_{k=1}^{K} D_k.
\] (10)

The boundary of the triangle \( D_k \) is formed by three straight lines called faces. By \( f_{ij} \) we denote a common face between two neighbouring elements \( D_i \) and \( D_j \). The symbol \( n_{ij} \) denotes the unit outer normal to \( D_i \) on the face \( f_{ij} \).

The solution \( \mathbf{q}(x,t) \) is approximated by \( \mathbf{q}_h(x,t) \) such that

\[
\mathbf{q}_h(x,t) \in C^1[0,T], \mathcal{V}_h(\Omega_h,T_h),
\] (11)

where the \( C^1([0,T], \mathcal{V}_h(\Omega_h,T_h)) \) is the Bochner space (see e.g. [5, 4]), \( \mathcal{V}_h(\Omega_h,T_h) \) is broken Sobolev space (see e.g. [4]) defined as

\[
\mathcal{V}_h(\Omega_h,T_h) = \{ \mathbf{u} \in L^2(\Omega_h); \mathbf{u}|_{D_k} \in \mathcal{P}_N(D_k), \forall D_k \in \mathcal{T}_h \},
\] (12)

and \( \mathcal{P}_N(D_k) \) is the space of all polynomials of degree \( \leq N \) on \( D_k \). If \( \mathbf{u} \in \mathcal{V}_h(\Omega_h,T_h) \), then \( \mathbf{u}|_{f_{ij}} \neq \mathbf{u}|_{f_{ji}} \) in general, where \( \mathbf{u}|_{f_{ij}} \) and \( \mathbf{u}|_{f_{ji}} \) denote the values of \( \mathbf{u} \) on \( f_{ij} \) considered from the interior of \( D_i \) and from the exterior of \( D_i \) (i.e. from \( D_j \)), respectively.

In order to derive the discrete problem, we use the following integral form of equation (6) (see, e.g. [9])

\[
\int_{D_k} \left( \frac{\partial \mathbf{q}_h}{\partial t} \ell - \frac{\partial \ell}{\partial x_1} \frac{\partial \mathbf{F}(\mathbf{q}_h)}{\partial x_1} - \frac{\partial \ell}{\partial x_2} \frac{\partial \mathbf{G}(\mathbf{q}_h)}{\partial x_2} \right) dx = - \int_{\partial D_k} \left( n_1 \frac{\partial \mathbf{F}(\mathbf{q}_h)}{\partial n} + n_2 \frac{\partial \mathbf{G}(\mathbf{q}_h)}{\partial n} \right)^* \ell d\mathbf{S}, \quad \forall \mathbf{D}_h \in \mathcal{T}_h,
\] (13)

which represents a weak form of the Euler’s equations in the sense of the broken Sobolev space \( \mathcal{V}_h(\Omega_h,T_h) \). In equation (13), the term \((n_1 \mathbf{F}(\mathbf{q}_h) + n_2 \mathbf{G}(\mathbf{q}_h))^* \) denotes a numerical flux, \( n_1, n_2 \) are the components of the unit outer normal \( \mathbf{n} \) on \( \partial D_k \). The symbol \( \ell \) denotes a test function taken from the space \( \mathcal{V}_h(\Omega_h,T_h) \), i.e., on \( D_k \) the test function \( \ell \) is from \( \mathcal{P}_N(D_k) \).

To evaluate the right hand side of the equation (13) we formally write for each face

\[
\int_{f_{ij}} (n_1 \mathbf{F}(\mathbf{q}_h) + n_2 \mathbf{G}(\mathbf{q}_h))^* \ell dx = \int_{f_{ij}} \mathbf{H}(\mathbf{q}_h|_{f_{ij}}, \mathbf{q}_h|_{f_{ji}}) \ell dx
\] (14)

where \( \mathbf{H}(\mathbf{q}_h|_{f_{ij}}, \mathbf{q}_h|_{f_{ji}}) \) is chosen numerical flux, see [5]. Here, we used the Harten-Lax-van Leer-contact (HLLC) flux of Toro et al. [13].

For the simplicity, let us denote the interior \( u|_{f_{ij}} \) and exterior \( u|_{f_{ji}} \) states on \( f_{ij} \) as \( u_L \) and \( u_R \), respectively. The computation of the HLLC flux is realized with the following steps:
1. Rotate momentum $\rho v_1, \rho v_2$ to face normal-tangent ($n,t$) coordinates for both the interior ($L$) and the exterior ($R$) states, i.e.

$$\begin{align*}
\rho v_n &= n_1 \rho v_1 + n_2 \rho v_2, \\
\rho v_t &= -n_2 \rho v_1 + n_1 \rho v_2,
\end{align*}$$

(15)

2. Compute the primitive variables $\rho, v_n, v_t, p, \text{ total energy } e$ and speed of sound $c$ and the state vector in face normal-tangent coordinates for both interior and exterior states, i.e., $\rho_{L/R}, v_{n,L/R}, v_{t,L/R}, p_{L/R}, e_{L/R}, c_{L/R}$, $q_{L/R} = [\rho_{L/R}, (\rho v_n)_{L/R}, (\rho v_t)_{L/R}, (\rho e)_{L/R}]$.

3. Compute wave speeds, i.e.

$$s_{L/R} = v_{n,L/R} \mp c_{L/R} V_{L/R},$$

(16)

where

$$V_{L/R} = \begin{cases} 1, & p^* \leq p_{L/R}, \\
\left[1 + \frac{\gamma+1}{2\gamma} \left(\frac{p^*}{p_{L/R}} - 1\right)\right], & p^* > p_{L/R},
\end{cases}$$

(17)

$$p^* = \frac{p_L + p_R}{2} + \frac{1}{8} (v_{n,L} - v_{n,R})(\rho_L + \rho_R)(c_L + c_R)$$

(18)

4. Compute HLLC flux, i.e.

$$H_n(q_L, q_R) = \begin{cases} F(q_L), & s_L > 0, \\
F(q_L) + s_L(q^*_L - q_L), & s_L \leq 0 \leq s^*, \\
F(q_R) + s_R(q^*_R - q_R), & s^* \leq 0 \leq s_R, \\
F(q_R), & s_R < 0,
\end{cases}$$

(19)

where the intermediate wave speed $s^*$ is given as

$$s^* = \frac{v_{n,L} + v_{n,R}}{2} + \frac{p_L - p_R}{(\rho_L + \rho_R)(c_L + c_R)}$$

(20)

and the intermediate states $q^*_{L/R}$ as

$$q^*_{L/R} = q_{L/R} \frac{s_{L/R} - v_{n,L/R} Q}{s_{L/R} - s^*}$$

(21)

where $Q$ is defined as

$$Q = \begin{bmatrix} 1 \\
1 \\
1 \\
1 \\
\end{bmatrix} \begin{bmatrix} s^* \\
\frac{p_{L/R}}{\rho_{L/R}(s_{L/R} - v_{n,L/R})} \\
\frac{p_{L/R}}{\rho_{L/R}(s_{L/R} - v_{n,L/R})} \\
\frac{p_{L/R}}{\rho_{L/R}(s_{L/R} - v_{n,L/R})} \\
\frac{p_{L/R}}{\rho_{L/R}(s_{L/R} - v_{n,L/R})} \\
\end{bmatrix}$$

(22)

5. Project flux back to the Cartesian coordinates, i.e.

$$H_1 = H_{n,1},$$

$$H_2 = n_1 H_{n,2} - n_2 H_{n,3},$$

$$H_3 = n_2 H_{n,2} + n_1 H_{n,3},$$

$$H_4 = H_{n,4}.$$  

(23)

The system (13) is solved by local time-marching method, the time derivative is replaced by the first order forward difference using a local value of the time step. For the cases of strong gradients presented in the numerical solution, we use generalized minmod limiter as is described in [15]. Furthermore, special treatment of boundary elements is adopted in case of a curved boundary for $N > 1$, see e.g. [9].
### 3.2 Finite Volume Method

For the finite volume approximation the computational domain $\Omega$ is again approximated by a polygonal domain $\Omega_h$ which is further decomposed into mutually disjoint control volumes $V_p$ with $p = 1, \ldots, P$, i.e.,

$$\Omega \approx \Omega_h = \bigcup_{p=1}^{P} V_p. \tag{24}$$

In this case $V_p$ can be an arbitrary polygon with its boundary $\partial V_p$ formed by a set of a straight lines called faces. The symbol $\mathcal{F}(p)$ denotes the set of all faces $f$ of the cell $V_p$.

The integral form of the system (6) is discretized in space using the collocated finite volume method (see e.g. [6]). This means that the solution $q(x, t)$ is approximated on each finite volume $V_p$ by

$$q_p(t) \approx \frac{1}{|V_p|} \int_{V_p} q(x, t) \, dx, \tag{25}$$

which leads to the semi-discrete form written as

$$|V_p| \frac{dq_p}{dt} = - \sum_{f \in \mathcal{F}(p)} (F_f^*, G_f^*) \cdot S_f = -R_p(q), \tag{26}$$

where $S_f$ is the outer normal vector with face area magnitude of the face $f \in \mathcal{F}(p)$ and $(F_f^*, G_f^*)$ is a numerical flux, see e.g. [5]. The numerical flux is realised using the HLLC flux and the piece-wise linear reconstruction is used (see e.g. [10]). The system (26) is solved by local time-marching method, the time derivative is replaced by the first order backward difference using a local value of the time step

$$|V_p| \frac{q_p^{n+1} - q_p^n}{\tau_p} = -R_p(q^{n+1}) \approx -R_p(q^n) - \frac{\partial R_p(q^n)}{\partial q_p} (q_p^{n+1} - q_p^n), \tag{27}$$

where $\frac{\partial R_p(q^n)}{\partial q_p}$ is the Jacobi matrix of $R_p$. The Jacobi matrix is approximated with matrix-free lower-upper symmetric Gauss-Seidel (LU-SGS) method, see [2, 7]. The finite volume solver was implemented and tested within OpenFOAM package by the author of the paper [7].

### 4 Numerical Results

Two test cases are considered, first is the Ringleb’s flow with known exact solution and the second is well-known case of the flow through channel with a bump (GAMM channel) which serves for the comparison of a numerical solutions of DGM and FVM.

#### 4.1 Ringleb’s Flow

This case considers transonic Ringleb’s flow. Ringleb’s flow is an exact solution to the Euler’s equations for $\gamma = 1.4$ obtained by Ringleb in 1940, see e.g. [11]. The subsonic inlet is assumed: the stagnation pressure $p_0 = 100 000$, the stagnation density $\rho_0 = 1.1684$, and the inlet angle $\alpha_I = \alpha(x), x \in \Gamma_I$, obtained from exact solution (see e.g. [12]) and outlet is assumed to be subsonic: the pressure $p_O = p(x), x \in \Gamma_O$, obtained from exact solution. Walls are treated with reflective boundary condition. See Figure 1 for details of the computational domain and the sketch of the computational grid. Figure 2 shows the contours of density $\rho$ for different order of polynomials $N = 1, 2, 4, 10$. One can see tremendous improvement between the numerical solution for $N = 1$ and the numerical solution for $N = 2$ and further improvement for $N = 4$, for $N = 10$ the numerical solution is very close to the exact solution.
Figure 1: (Left) Sketch of the computational domain for Ringleb’s flow, generated such that whole domain is subsonic. Parameters used to generate computational domain are non-dimensional inlet velocity magnitude $0.1$, and minimum and maximum of the stream function $1.5$ and $3.0$, respectively, see e.g. [12, 11]. (Right) Sketch of the computational grid.

Figure 2: Details of contours of density $\rho$ for different order of polynomials. Black lines indicate exact solution and the white lines indicate the numerical solution.

4.2 Transonic Flow Through GAMM Channel

As the second test case we choose the transonic flow through the two-dimensional test channel with a bump, i.e., the so-called Ron-Ho-Ni or also GAMM channel, see Figure 3. This test case was solved by many researchers, see e.g. [7]. At the subsonic inlet we prescribe the stagnation pressure $p_0 = 100 000$, the stagnation density $\rho_0 = 1$, and the inlet angle $\alpha_I = 0$. At the subsonic outlet we kept the pressure $p_O = 737 000$. The upper and lower parts are solid walls.

Two computational triangular grids were used, first $G1$ with approximately 6 000 elements and second $G2$ with approximately 24 000 elements. All DGM data presented in this subsection are for $N = 1$, i.e., linear elements. Figure 4 shows the distribution of Mach number along the lower wall. Both methods show similar results, however the FVM performs better in a whole domain as can be seen on Figure 5. The main reason for worse performance of DGM lies in the problem that minmod limiter introduce too much numerical diffusion as can be seen in Figure 4 approximately after the shock, i.e., $x \in (2, 3)$, where the solution of the FVM is closer to isentropic Mach number.
Figure 3: Sketch of the computational domain for the GAMM channel.

Figure 4: Mach number distribution along the lower wall. Black line serves as a reference solution obtained by FVM with very fine grid.

Figure 5: Mach number contours. (Upper) G1. (Lower) G2. Red lines indicate the DGM, blue lines indicate the FVM solutions. Black lines serve as a reference solution obtained by FVM with very fine grid.

5 Conclusion

In this contribution two methods for the numerical approximation of the gas dynamic problems were investigated, namely OpenFOAM’s implementation of finite volume method and in-house implementation of discontinuous Galerkin method in Julia programming language (see [1]). The high-order accuracy DGM shows promising results on smooth data (Ringleb’s flow). However, in
case of non-smooth data the DGM combined with minmod limiter shows unsatisfactory behav-
ior. The minmod limiter introduce too much numerical diffusion and completely destroys DGM
solutions for $N > 1$.

Acknowledgment

This work was supported by the Grant Agency of the Czech Technical University in Prague, grant SGS
2019. Authors acknowledge support from the EU Operational Programme Research, Development and
Education, and from the Center of Advanced Aerospace Technology (CZ.02.1.01/0.0/0.0/16_019/0000826),
Faculty of Mechanical Engineering, Czech Technical University in Prague.

References