DEVELOPMENT OF NON-REFLECTIVE BOUNDARY CONDITION FOR FREE-SURFACE FLOWS

J. Fürst 1, J. Musil 1

1 Department of Technical Mathematics, Faculty of Mechanical Engineering, Czech Technical University in Prague, Karlovo nám. 13, 121 35 Praha 2, Czech Republic

Abstract

The article presents development of non-reflective boundary condition for free-surface flows and its implementation in the framework of volume of fluid (VOF) advection method for Navier-Stokes equations in the finite-volume CFD toolbox OpenFOAM. Derivation of the boundary condition is based on simplified 1D shallow water equations as presented in [1].

Keywords: OpenFOAM, volume of fluid, VOF, shallow water, non-reflective, boundary condition, finite volume method.

1 Introduction

A frequently encountered problem in scientific computing is the design of artificial boundaries. The goal is to limit a computational domain to keep the number of cells within reasonable bounds yet still end up with a solution that approximates the correct result for an unbounded domain. This kind of boundaries, sometimes called absorbing or partially reflecting boundaries, allows disturbances generated within the solution domain to pass through the model boundary unhampered, while information from outside the solution domain is simultaneously specified to achieve the desired interior solution [1]. In the case of shallow water equations this means that there can be prescribed water height and wave velocity on the boundary with water waves passing out unhampered. In some flow regimes, Navier-Stokes equations describe same phenomenon as shallow water equations and therefore we use boundary conditions motivated by theory of shallow water equations. An original motivation of this work was simulation of vertical water pump situated in semi-opened basin, see Fig. 1. The simulation was performed in OpenFOAM [2], where lack of sufficient boundary conditions led to reflecting water waves back to computational domain and due to further wave interactions, depending on geometry of the case, non-physical solution was obtained.

Section 2 presents some details of derivation of non-reflective boundary conditions for shallow-water equations, see original article [1], with results of numerical simulation of 1D dam break problem, described by shallow water equations.

Section 3 describes implementation of these boundary conditions in the OpenFOAM package for VOF [3] method, with results of numerical simulation of 2D dam break problem and 3D water pump problem, described by Navier-Stokes equations in VOF formulation.

Figure 1: Vertical water pump simulation scheme
2 Boundary conditions for shallow water equations

2.1 Derivation of boundary conditions

With assumption, that the horizontal length scale of computational domain is much greater than the vertical, fluid is inviscid and incompressible, no sources of fluid are present and bed level is constant, 1D shallow water equations can be written in following simple form

\[ \frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0 \] (1)

\[ \frac{\partial (uh)}{\partial t} + \frac{\partial (u^2h + \frac{1}{2}gh^2)}{\partial x} = 0 \] (2)

where \( h(x,t) \) denotes the water surface height, \( u(x,t) \) is the water wave velocity and \( g = \text{const.} \) is the gravitational acceleration. The equation (1) resp. (2) states for mass resp. momentum conservation. If \( u, h \) are smooth, equations (1),(2) can be written in primitive variables.

\[
\begin{pmatrix}
    u \\
    h
\end{pmatrix}_t + \begin{pmatrix}
    u & h \\
    g & u
\end{pmatrix}_x \begin{pmatrix}
    u \\
    h
\end{pmatrix} = \begin{pmatrix}
    0 \\
    0
\end{pmatrix}
\] (3)

In order to decompose these equations into the system of independent equations, one uses the so called Riemann variables, or Riemann invariants, and after some algebraic operations the original equations (1), (2) become

\[ R_1(u,h)_t + \lambda_1(u,h) \cdot R_1(u,h)_x = 0 \] (4)

\[ R_2(u,h)_t + \lambda_2(u,h) \cdot R_2(u,h)_x = 0 \] (5)

which are two separate hyperbolic partial differential equations, where Riemann variables are \( R_{1,2}(u,h) = u \pm 2\sqrt{gh} \) with \( \lambda_{1,2}(u,h) = u \pm \sqrt{gh} \) representing characteristic advection speeds.

Further, analytic solution by the method of characteristics gives

\[ R_i(u(x,t), h(x,t)) = R_i(x,t) = R^0_i(x - \lambda_i(x,t) \cdot t) \] \[ i = 1, 2 \] (6)

where \( R^0_i(x) = R_i(x,0) \) is initial condition. Solution of the problem in terms of original variables \( h(x,t) \) and \( u(x,t) \) is given as combination of Riemann invariants.

\[ u = \frac{1}{2}(R_1 + R_2) \quad h = \frac{1}{16g}(R_1 - R_2)^2 \] (7)

Now, 1D computational domain is considered, with artificial boundary on the left \( (x = x_B = 0) \). If the flow regime is considered sub-critical \( (|u| < \sqrt{gh}) \), then following conditions hold for characteristic speeds: \( \lambda_1 > 0 \) and \( \lambda_2 < 0 \). So throughout derivation of non-reflective boundary condition we can use following scheme

![Figure 2: Sketch of left (artificial) boundary of Ω](image-url)
where Ω denotes computational domain, ∂Ω its boundary, \(x_B\) represents coordinate of left boundary, \(x_1\) is coordinate of first finite-volume-cell center and \(x_{-\infty}\) is coordinate “far” from computational domain.

Now if equation (6) is used, one get relations for Riemann variables on the left and right side of the boundary, where, for further purposes of numerical simulation, initial condition is taken as \(R_i(x,t)\) and solution in next time step \(R_i(x,t+\Delta t)\) is being looked for. Calm surface is assumed outside the Ω (on the left side), so \(u = u_{-\infty} = 0\) and \(h = h_{-\infty} \neq 0\) are considered here, see equation (8).

\[
R_1(x_B, t + \Delta t) = 0 + 2\sqrt{gh_{-\infty}} = \text{const.} \tag{8}
\]

Inside the Ω holds the implicit relationship given by following equation,

\[
R_2(x_B, t + \Delta t) = R_2(x_B - \lambda_2(x_\xi, t) \cdot \Delta t, t), \quad x_\xi = x_B - \lambda_2(x_\xi, t) \cdot \Delta t \tag{9}
\]

where \(x_\xi \in (x_B, x_1)\). To resolve equation (9) the following approximation was made.

\[
R_2(x_B, t + \Delta t) = R_2(x_B - \lambda_2(x_\xi, t) \cdot \Delta t, t) = R_2(x_1, t) + O(|x_1 - x_B|) \tag{10}
\]

Original variables on the artificial boundary are given by using equations (7), so finally we have

\[
u(x_B + \Delta t) = \frac{1}{2} \left(2\sqrt{gh_{-\infty}} + u(x_1, t) - 2\sqrt{gh(x_1, t)}\right) \tag{11}
\]

\[
h(x_B + \Delta t) = \frac{1}{16g} \left(2\sqrt{gh_{-\infty}} - u(x_1, t) + 2\sqrt{gh(x_1, t)}\right)^2. \tag{12}
\]

One must keep in mind, that these terms are valid only for sub-critical advection speeds \(|u| < \sqrt{gh}\).

### 2.2 Numerical results

Performance of boundary condition was tested on 1D dam break simulation. The non-reflective boundary condition was prescribed at the left boundary \((x = 0)\) whereas a combination of Dirichlet condition for the velocity \(u = 0\) and Neumann condition for the volume fraction \(\frac{\partial h}{\partial \vec{n}} = 0\) was prescribed at the right boundary. The numerical solution of equations (1), (2) was obtained using the explicit Euler method in time and the first order finite volume scheme with the HLL numerical flux. \[4\]
3 Implementation of new boundary condition to OpenFOAM

3.1 Implementation

As mentioned above, the main goal of this work was to develop a non-reflective boundary condition for free-surface flows described by the system of Navier-Stokes equations in the so-called volume of fluid formulation (VOF). The problem is described by the following system of equations

\[
\frac{\partial u_j}{\partial x_j} = 0, \quad \tag{13}
\]

\[
\frac{\partial \rho u_i}{\partial t} + \frac{\partial (\rho u_i u_j)}{\partial x_j} + \frac{\partial p}{\partial x_i} = \frac{\partial \tau_{ij}}{\partial x_j} + \rho g_i, \quad \tag{14}
\]

\[
\frac{\partial \alpha}{\partial t} + \frac{\partial (\alpha u_j)}{\partial x_j} = 0. \quad \tag{15}
\]

Here \(0 \leq \alpha \leq 1\) is the liquid fraction in the mixture (\(\alpha = 0\) corresponds to air, \(\alpha = 1\) corresponds to water) and the density of the mixture is \(\rho = \alpha \rho_{\text{water}} + (1 - \alpha) \rho_{\text{air}}\). The symmetric tensor \(\tau_{ij}\) expresses tangential stresses and \(g_i\) is the acceleration due to gravity [3]. The description of the boundary condition is given here in general three-dimensional case, see Fig. 4. The two-dimensional problem can be treated by assuming one cell in \(z\)-direction.

In order to construct the non-reflective boundary conditions for the left boundary (red rectangle in Fig. 4) one uses terms (11) and (12) from previous section. The task here is to determine \(h(x_1, t)\)
and \( u(x_1, t) \), which is done as follows\(^1\)

\[
\begin{align*}
    h(x_1, t) &= \frac{\int_W \alpha(x_1, y, z, t) \, dy \, dz}{W} \\
    u(x_1, t) &= \frac{\int_W \alpha(x_1, y, z, t) \cdot u_x(x_1, y, z, t) \, dy \, dz}{\int \alpha(x_1, y, z, t) \, dy \, dz}
\end{align*}
\]  

(16)

where \( W \) denotes area of the left boundary. Note, that this implementation eliminates the dependency upon the variable \( z \) by taking mean values in this direction. Now \( u(x_B, t + \Delta t) \) and \( h(x_B, t + \Delta t) \) can be evaluated. Boundary condition for \( \alpha \) is then computed as

\[
\alpha(x_B, y, z, t + \Delta t) = \begin{cases} 
0 & y > h(x_B, t + \Delta t) \\
1 & y < h(x_B, t + \Delta t)
\end{cases}
\]

Boundary condition for \( \vec{u} \) is set as \( \vec{u}(x_B, y, z, t + \Delta t) = (u(x_B, t + \Delta t), 0, 0) \) and for pressure \( \frac{\partial p}{\partial \vec{n}} = 0 \).

3.2 Numerical results

As first (testing) case, the dam break problem solved in two-dimensional domain with initial condition set to \( \vec{u} = \vec{0} \) and \( \alpha \) according to Fig. 5(a) was chosen. At the upper, right, and lower boundary the non-slip boundary condition is used for \( \vec{u} \) and homogeneous Neumann condition for \( \alpha \) and \( p - h \rho g \). Two variants of boundary conditions were tested at the artificial left boundary condition, the standard OpenFOAM condition allowing variable water level (i.e. \( \frac{\partial \alpha}{\partial n} = 0 \) combined with prescribed volumetric flow rate) and the new non-reflective boundary condition.

![Figure 5: VOF simulation of dam break problem](image)

(a) Initial condition, \((t = 0s)\)

(b) Variable height boundary condition, \((t = 2s)\)

(c) Non-reflective boundary condition, \((t = 2s)\)

Figures 5(b) and 5(c) show the water fraction in time \( t = 2s \) obtained with the OpenFOAM solver using two above mentioned boundary condition. One can see that the solution obtained with the first boundary condition is spoiled by false reflection from the boundary. On the other hand the non-reflective boundary condition is transparent to incoming waves. At Fig. 6 original solution (upper picture) corresponds very closely to a solution obtained on an elongated domain (bottom picture).

\(^1\) This approach requires a structured grid next to the boundary
Figure 6: Non-reflective boundary condition on original and elongated domain, \((t = 5\, s)\)

Second case demonstrates the performance of the non-reflective boundary condition in the case of 3D VOF simulation of vertical water pump in a semi-opened basin. We assume either the new non-reflective or the variable height boundary condition at the left boundary \((x = x_{\text{min}})\), the non-slip boundary conditions at other vertical walls and at the bottom of the basin, and a prescribed flow rate of \(4\ m^3/s\) at the tube outlet. The Fig. 8 shows the comparison of standard boundary conditions with non-reflective on left side of basin. The initial condition is set according the Fig. 7. The figures in the right column of Fig. 8 show non-physical behavior due to waves being reflected back to computational domain. Although the non-reflective boundary condition effectively removes the reflection of large size waves (compare the left and right columns in the Fig. 8), there are still some small wavelets near the left boundary (see Fig. 8, last row). There might be several reasons for this phenomenon. At first, we are using boundary conditions derived from 1D shallow water equations for solving a 3D VOF case, i.e. we prescribe constant \(\vec{u}\) and \(\alpha\) along basin’s width (z-direction) and therefore we neglect spatial character of waves. Second problem might be the violating the flow regime, which should be close to shallow water (i.e. not satisfying low vertical velocity on boundary). Another problem could be first order spatial accuracy of non-reflective boundary condition.
Figure 7: VOF simulation of water pump, initial condition

Figure 8: VOF simulation of water pump, the non-reflective boundary condition in the left column, the variable height condition in the right column.
4 Conclusion

The development of non-reflective boundary conditions for free-surface flows described by Navier-Stokes equations in the volume of fluid formulation (VOF) was presented in this paper. Numerical results of a dam break problem demonstrate the performance of this boundary condition. Next, the water pump case shows that the boundary condition can be effectively used also in the case of practical 3D cases of two-phase flows.

Acknowledgment

This work was supported by the Grant Agency of the Czech Technical University in Prague, grant No. SGS16/206/OHK2/3T/12.

References


