NONLINEAR INSTABILITY ANALYSIS OF POWER-LAW SHEETS

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Abstract

Research on nonlinear instability of Power-law plane sheets has been conducted using Carreau law. Combined with asymptotic expansion and long wave assumption, the governing equations and boundary conditions were performed using integral transform. The first-order dimensionless dispersion relation between unstable growth rate and wavenumber was obtained and the second-order interface disturbance amplitude was calculated. By comparison and analysis of components of the second-order interface disturbance amplitude, the effects of power-law index $n$ ($n<1$) were investigated and the condition under which the shear-thinning effect can be evident was concluded, thus contributing to theoretical basis and technological means to improve atomization of power-law sheets.

Key words: Carreau law, nonlinear stability, shear-thinning, asymptotic expansion

1 Introduction

In recent years, gel propellant has been widely used in the rocket engine application, with better performance than the traditional liquid propellant and solid propellant. The gel propellant is a time-dependent non-Newtonian fluid and usually has the properties of shear thinning [1]. However, due to its high viscosity and rheological properties, it is difficult for the gel propellant to atomize, and its atomization mechanism has not been fully recognized, which is the main bottleneck to restrict the performance of gel rocket engine. Therefore, it is of great significance to study the fracture behavior of non-Newtonian fluids.


In engineering, when taking a test about steady shear rheological behavior on gel propellant, the relationship between shear viscosity $\mu$ and shear rate $\dot{\gamma}$ can be fitted as $\mu = k \dot{\gamma}^{n-1}$, therefore this kind of fluid is usually called power-law fluid. Up to now, reports on atomization mechanism of power-law fluids are rare, most of which are focused on flow stability for power-law fluid films along the inclined plane. Lin and Hwang [16] used the method of normal mode to study linear stability for power-law liquid films flow down an inclined plane. The results showed that shear-thinning liquid films along the inclined plane are more unstable than Newtonian’s, and the dimensional wave speed of shear-thinning liquid is faster than that of the Newtonian’s. Balmforth \textit{et al.} [17] found superposed layers of fluid flowing down an inclined plane are prone to interfacial instability even in the limit of zero Reynolds number. They presented two versions of lubrication theory for superposed layers of non-Newtonian fluid with power-law rheology and carried on both linear analysis and nonlinear numerical simulation. Amaouche \textit{et al.} [18] conducted modeling and linear stability for a power-law fluid film flowing down an inclined plane with the method of lubrication theory and weighted residual approach. Also, they performed an incomplete regularization procedure to cure the rapid divergence of the reduced two-equation model. Jawadi \textit{et al.} [19] proposed an asymptotic numerical method to solve the power-law flow. Comparisons with the numerical
results found in the literature demonstrate the efficiency of the proposed numerical method. Ruyer-Quil et al. [20] studied wavy regime of a power-law fluid flowing down an inclined plane under the action of gravity, within the frame of the lubrication approximation by means of the weighted residual approach.

Considering rare researches on stability of free power-law liquid jets or sheets, in this paper, planar power-law liquid sheet with symmetric disturbances has been investigated on its nonlinear stability. Both first-order dispersion equation and second-order interface disturbance are obtained. And influence of power-law behavior on sheet instability has also been studied.

2 Theoretical Model

Imagine a uniform infinite free sheet for power-law fluids surrounded by stationary inviscid incompressible irrotational gas. For the liquid sheet, the density, thickness, coefficient of surface tension, power-law index, and velocity are respectively denoted by \( \rho \), \( 2a \), \( \sigma \), \( n \) (<1), and \( U \). While for the gas, \( \rho_g \) and \( p_g \) represent its density and pressure, respectively, as is shown in Figure 1. \( \eta_j \) indicates the disturbance amplitude of the upper and lower surfaces: when \( j=1 \), it corresponds to the upper surface; when \( j=2 \), it corresponds to the lower surface. Here, the influence of the gravitational and magnetic field is ignored.

![Figure 1: Schematic diagram of free power-law sheets under symmetric disturbances](image)

2.1 Liquid Phase

The governing equations for the liquid phase are

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]

\[
\rho(u_{\text{in}} + uu_x + vu_y) = -p_x + \frac{\partial}{\partial x} \tau_{xy} + \frac{\partial}{\partial y} \tau_{yy}
\]

\[
\rho(v_{\text{in}} + uv_x + vv_y) = -p_y + \frac{\partial}{\partial x} \tau_{xy} + \frac{\partial}{\partial y} \tau_{yy}
\]

In the situation here, we assume that the sheet thickness is far less than the wavelength. Using the long-wave assumption, let the ratio of the sheet thickness and the wavelength satisfy: \( \frac{a}{L} = \varepsilon \ll 1 \). With the selected characteristic wavelength \( L \), characteristic velocity \( U \) and characteristic viscosity \( \mu \), the dimensionless form of each parameter is:

\[
y = a\hat{y}, \quad x = L\hat{x}, \quad \eta = a\hat{\eta}, \quad u = U\hat{u}, \quad v = \varepsilon U\hat{v}, \quad t = \frac{L}{U},
\]

\[
p = \mu \frac{U}{L} \hat{p}, \quad \tau = \mu \frac{U}{L} \hat{\tau}, \quad \gamma = \frac{U}{L}, \quad \phi = \varepsilon UL\hat{\phi}, \quad \rho = \frac{\rho_f}{\rho}, \quad Re = \frac{\rho UL}{\mu}, \quad We = \frac{\rho U^2 L}{\sigma}, \quad P_e = \frac{\rho e L}{\mu U}.
\]

Here, the subscript ‘g’ denotes the corresponding parameters for the gas.

Then the governing equations can be nondimensionalized as follows:

\[
u_{\text{in}} + v_{\text{in}} = 0
\]

\[
Re(u_{\text{in}} + uu_x + vu_y) = -p_{\text{in}} + \frac{\partial}{\partial x} \tau_{xy} + \frac{1}{\varepsilon} \frac{\partial}{\partial y} \tau_{yy}
\]

\[
\varepsilon^2 Re(v_{\text{in}} + uv_x + vv_y) = -p_{\text{in}} + \frac{\varepsilon}{\partial x} \tau_{xy} + \frac{\partial}{\partial y} \tau_{yy}
\]
It is easy to infer that \( u \) can be written in the following form when making the stress component satisfy \( \tau_{xy} = \varepsilon T_{xy} \) (here, \( \tau_{xy} \) is of order \( \varepsilon \), while \( T_{xy} \) is of order 1).

\[
\begin{align*}
u = u_0(x, t) + \varepsilon^2 u_2(x, y, t) + \cdots
\end{align*}
\]

(7)

For the varicose mode, it should satisfy:

\[
v = 0 \quad \text{at} \quad y = 0
\]

(8)

The kinematic boundary condition is

\[
\eta_{jy} + u \eta_{jy} = v \quad \text{at} \quad y = (-1)^{j+1} + \eta_j
\]

(9)

where, \( j=1 \) denotes the upper surface, and \( j=2 \) denotes the lower surface.

Combining the equations (4) (7) (8) with (9) gives

\[
\eta_{jy} + u \eta_{jy} = -\eta_j + 1 \quad \text{at} \quad y = (-1)^{j+1} + \eta_j
\]

(10)

The momentum conservation equations (5) and (6) can be rewritten as:

\[
\begin{align*}
p + \tau_{xx} &= f(x, t) \\
Re(u_{ii} + u_{ii} u_{ii}) &= 2 \partial_x \tau_{xx} - s + \partial_y T_{xy}
\end{align*}
\]

(11)

Integrating on both sides of the upper equations for \( y \) from 0 to 1, there is

\[
\begin{align*}
Re(1 + \eta_j)(u_{ii} + u_{ii} u_{ii}) &= \partial_x [2(1 + \eta_j) \tau_{xx} | - f(1 + \eta_j) + T_{xy} | y = \eta_j - \eta_j | y = 0]
\end{align*}
\]

(13)

The normal dynamic boundary condition and the tangential dynamic boundary condition are respectively:

\[
\begin{align*}
\tau_{xy} + p &= P_g - \Gamma \eta_{jy} \quad \text{at} \quad y = (-1)^{j+1} + \eta_j \\
T_{xy} &= 2 \eta_j \tau_{xx} \quad \text{at} \quad y = (-1)^{j+1} + \eta_j
\end{align*}
\]

(14)

(15)

where, \( \Gamma = \frac{\varepsilon \sigma}{\mu U} \), reflects the relative influence of surface tension and viscous force on the liquid / gas interface.

### 2.2 Gas Phase

Since the gas is inviscid, incompressible and irrotational, the potential function \( \phi_g \) of the gas should satisfy

\[
\phi_{gx} + \phi_{gry} = 0
\]

(16)

Then the solution is

\[
\hat{\phi}_g = (-1)^{j+1} A e^{-i k y}
\]

(17)

where, \( A \) is a constant, \( k \) is the wavenumber. \( \hat{\phi}_g(k, y) \) is the Fourier transform function of \( \phi_g(x, y) \), that is

\[
\hat{\phi}_g(k, y) = F[\phi_g(x, y)] = \int_{-\infty}^{\infty} e^{-ikx} \phi_g dx
\]

(18)

At \( y = (-1)^{j+1} + \eta_j \), the gas should meet the following boundary condition:

\[
\eta_{ji} = \phi_{gj}
\]

(19)

\[
P_g + \Re \{ P_g \} (x, y) = 0
\]

(20)

With the Equations (17)(18)(19)(20), there is

\[
P_g = (-1)^j \Re \{ P_g \} \eta_{ji}
\]

(21)

here, \( \Re \{ \} \) indicates the Hilbert transform of corresponding variables.
Combining the equations of the liquid phase with those of the gas phase, a two-variable ($u$ and $\eta_j$) system of equations can be obtained finally

\[ \eta_j + \partial_t [(1 + \eta_j)u] = 0 \]  
\[ Re(1 + \eta_j)u_t + u_{xx} = \partial_x [2(1 + \eta_j)\tau_{xx} - (1 + \eta_j)P_{gs} + (1 + \eta_j)\Gamma \eta_j] \]  

where, $P_{gs} = (-1)^{1/2} Re \mathcal{H} \eta_j$.  

The non-dimensional Carreau model is adopted as the constitutive model:

\[ \mu_{eff}(\dot{\gamma}) = [1 + (\dot{\gamma}/\dot{\gamma_c})^n]^{-(\eta_s - 1)/n} \]  

where, $\dot{\gamma_c}$ denotes the critical shear rate of the power-law sheet.

Assuming all the disturbance parameters can be expanded into a power-law series of small initial amplitude $\eta_0$, thus there is

\[ (u, \tau_{xx}, \eta_j, P_{gs}) = (1, 0, 0, 0) + \varepsilon(u_1, \tau_{1xx}, \eta_{1j}, P_{gs1}) + \varepsilon^2(u_{2xx}, \tau_{2xx}, \eta_{2j}, P_{gs2}) + \cdots \]  

Therefore,

\[ \tau_{xx} = 2u_1[1 + (2u_1/\dot{\gamma}_c)^n]^{\eta_s - 1/2} = 2(\varepsilon u_{1xx} + \varepsilon^2 u_{2xx})[1 + \frac{4\varepsilon^2(u_{1xx} + u_{2xx})^2}{\dot{\gamma}_c^2}]^{\eta_s - 1/2} \]  

If $\dot{\gamma}_c = O(1)$, it needs to proceed to $O(\varepsilon^1)$ so as to display the influence of the power-law index $n$;  
If $\dot{\gamma}_c = O(\varepsilon)$, let $\dot{\gamma}_c = \varepsilon^{1/2}$, then $\tau_{xx} = 2(\varepsilon u_{1xx} + \varepsilon^2 u_{2xx})[1 + \frac{4\varepsilon^2 u_{1xx}^2}{\varepsilon^{1/2}} + O(\varepsilon^3)]^{\eta_s - 1/2}$, in this case it is obvious that the term of $O(\varepsilon)$ is nonlinear, so linear analysis cannot be done under this circumstance. Therefore, it needs to be modified. Let $\dot{\gamma}_c = O(\varepsilon^{1/2})$ and $\dot{\gamma}_c = \varepsilon^{3/2}$, then

\[ \tau_{xx} = 2(\varepsilon u_{1xx} + \varepsilon^2 u_{2xx})[1 + \frac{4\varepsilon^2 u_{1xx}^2}{\varepsilon^{3/2}} + \cdots]^{\eta_s - 1/2} \]  

Further there are

\[ \tau_{1xx} = 2u_1 \]  
\[ \tau_{2xx} = 2u_{1xx} \frac{4u_{1xx}^2}{\Gamma_1^{1/2}} - \frac{n - 1}{2} + 2u_{2xx} = 4(n - 1) \frac{u_{1xx}^3}{\Gamma_1^{1/2}} + 2u_{2xx} \]  

### 3 Perturbation Theory

#### 3.1 First order

The first-order equations can be written in the following form:

\[ \eta_{1j} + \eta_{1j} + u_{1i} = 0 \]  
\[ Re(u_{1i} + u_{1i}) = 2\partial_x \tau_{1xx} + \bar{\rho} Re \mathcal{H} \eta_{1j} \]  

Assume the solution of the first-order equations can be written in the following form:

\[ (u_1, \eta_{1j}) = [\hat{u}_1, \hat{\eta}_1] e^{i(k_1 x - \omega t)} + c.c \]  

thus

\[ \hat{u}_1 = \frac{\alpha k - k_1}{k_1} \hat{\eta}_1 \]  
\[ \frac{Re(\omega^2 - k_1^2)}{k_1} = 4k_1(\omega - k_1) + \bar{\rho} \alpha k_1 \]  
\[ Re(\omega - k_1) = 4k_1(\omega - k_1) + \bar{\rho} \alpha \omega \]
### 3.2 Second order

The second-order equations are

\[
\eta_{2,1} + \eta_{2,2} + u_{2,1} + \eta_{1,1} = 0 \tag{35}
\]

\[
\text{Re}(u_{2,1} + \eta_{1,2} + \eta_{2,2}) = 2\eta_{1,2} + 2\eta_{2,2} + 2\eta_{1,2} + \partial \eta_{2,1} + \partial \eta_{1,1} + \Gamma_{\eta,2} + \Gamma_{\eta,1} \tag{36}
\]

The initial condition is given as

\[
\eta_{0,0} = -\eta_0 \cos k_x x = \frac{1}{2} \eta_0 \exp (ik_x x) + c.c \tag{37}
\]

The solution of the second-order equations can be assumed as

\[
\eta_{1,1} = \hat{\eta}_{1,1} e^{2(k_x t - \omega_k t)} + \hat{\eta}_{2,2} e^{2(k_x t - \omega_k t)} + \hat{\eta}_{1,2} e^{i(k_x t - \omega_k t)} + \hat{\eta}_{2,2} e^{i(k_x t - \omega_k t)} + \hat{\eta}_{1,2} e^{i(k_x t - \omega_k t)} + c.c \tag{38}
\]

Substituting Equations (37) (38) and the first-order solution into the second-order equations (35) (36), and merging terms containing the same \( \varepsilon \) index (\( e^{i(k_x t - \omega_k t)} \), \( e^{2(k_x t - \omega_k t)} \), \( e^{i(k_x t - \omega_k t)} \), \( e^{i(k_x t - \omega_k t)} \), the second-order solutions can be obtained as follows.

\[
\hat{\eta}_{21} = (-1)^{i+1} \frac{2iRe(\omega_1 - k_x^2) - 8k_x^2 (\omega_1 - k_x) - ik_x \omega_1 \varepsilon \text{Resgn}(\kappa_1) + ik_1 \Gamma \hat{\eta}_{11}^2}{2iRe(\omega_1 - k_x^2) - 16k_x^2 (\omega_1 - k_x) + 4ik_x \omega_1 \varepsilon \text{Resgn}(\kappa_1) - 8ik_1 \Gamma} \tag{39}
\]

\[
\hat{\eta}_{22} = (-1)^{i+1} \frac{8(n-1)k_x^3 (\omega_1 - k_x)}{\Gamma} \frac{12k_x^2 (\omega_1 - k_x) - iRe(\omega_1 - k_x)^2 + 9ik_1 \Gamma - 3i\omega_1 \varepsilon \text{Resgn}(k_1)}{\hat{\eta}_{22}} \tag{40}
\]

\[
\hat{\eta}_{23} = (-1)^{i+1} \frac{24(n-1)k_x^3 (\omega_1 - k_x)}{\Gamma} \frac{iRe \Lambda^2 + 4ik_x \Lambda + ik_1 (3\omega_1 - i\omega_1)}{\text{Resgn}(k_1) + ik_1 \Gamma} \tag{41}
\]

where, \( D_{\omega}(\omega, k) = \varepsilon \omega^2 - \frac{k^3}{We} + \frac{(\omega_1 - k_x)^2}{k} + \frac{4ik_x \omega_1 \varepsilon \text{Resgn}(k_1)}{Re} \), \( \Lambda = 3\omega_1 - i\omega_1 + ik_k \).

Hence, the second-order disturbance amplitude can be written as

\[
\eta_{2,1} = \hat{\eta}_{21} e^{2(k_x t - \omega_k t)} + \hat{\eta}_{22} e^{2(k_x t - \omega_k t)} + \hat{\eta}_{23} e^{i(k_x t - \omega_k t)} + \hat{\eta}_{24} e^{i(k_x t - \omega_k t)} + c.c \tag{42}
\]
4 Results and Discussions

From the above expressions, it can be inferred that:

1. When $\dot{\gamma} (2\alpha, 2k_2) = 0$, there would be singularities appearing in the solution of $\hat{\eta}_{21}$.

2. For the Carreau liquid sheets studied here, the critical shear rate $\dot{\gamma}_c$ should be as small as possible. Since $\dot{\gamma}_c = e^{\psi_{\alpha} \Gamma_{\nu_2}}$, $\Gamma_{\nu_2}$ should also be as small as possible. When $\Gamma_{\nu_2}$ is small enough, the effect of shear-thinning would become evident.

3. The final expression of the second-order disturbance amplitude implies that the power-law factor only impacts on instability of waves with one time and three times the wavelength.

In the following, the influence of power-law characteristics on liquid sheet rupture is further analyzed by waveform diagram. Figure 2 illustrates evolution of waveform for power-law liquid sheets, with other parameters fixed at $Re=1000$, $We=553.4$, $\bar{p}=0.0012$, $\eta_0=0.01$, $\Gamma_{\nu_2}=0.0001$, $k_1=0.48$, $n=0.4$. It can be seen that as the dimensionless time increases, the amplitude of surface wave would increase, with the upper surface and lower surface staying symmetric. At $t=250$, there appears obvious distortion on the surface wave, which manifests the influence of shear-thinning. At $t=295.75$, the upper and lower surface touches each other, and the sheet would break up into lots of liquid filaments.

![Waveform Diagram](image)

Figure 2: Evolution of waveform for power-law liquid sheets ($Re=1000$, $We=553.4$, $\bar{p}=0.0012$, $\eta_0=0.01$, $\Gamma_{\nu_2}=0.0001$, $k_1=0.48$, $n=0.4$)

The effect of the power-law index $n$ on the ratio of the second-order power-law and non-power-law amplitude intensity $\eta (\eta = \left| \frac{\hat{\eta}_{21} + \hat{\eta}_{24}}{\hat{\eta}_{21}} e^{\alpha t} \right|$) is plotted in Figure 3. It can be obtained that when $\Gamma_{\nu_2}$ is
fixed, with the increase of the power-law index $n$, the power-law property becomes weaker, and instability caused by the non-power-law amplitude would become dominant. While $n$ is fixed, if double the value of $\Gamma^{1/2}$, then the ratio of the second-order power-law and non-power-law amplitude $\eta$ would reduce to $1/4$ of the original, and the corresponding power-law instability would be weakened.

![Figure 3: Effect of the power-law index $n$ on the ratio of the second-order power-law and non-power-law amplitude intensity $\eta$](image)

The variation of the ratio of the second-order power-law and the non-power-law amplitude intensity $\eta$ versus the wave number $k$ at different times is shown in Figure 4. It can be seen that with time going on, the property of the second-order amplitude at the gas/liquid interface is more and more obvious, which means the effect of shear-thinning is more and more evident. The stage when the liquid sheet is close to rupture is actually a strongly nonlinear process.

![Figure 4: Variation of the ratio of the second-order power-law and the non-power-law amplitude intensity $\eta$ versus the wave number $k$ at different times](image)

**References**


