NOTE ON THE USE OF CAMASSA–HOLM EQUATIONS FOR SIMULATION OF INCOMPRESSIBLE FLUID TURBULENCE

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Abstract

The aim of this short communication is to briefly introduce the Camassa–Holm equations as a working model for simulation of incompressible fluid turbulence. In particular we discuss its application for turbulent boundary layer flows. This model (and related models) is studied for several years in mathematical community, starting from Leray [23]. It can be understood as a generalization of some classical fluid models (Navier-Stokes equations, Prandtl boundary layer equations), showing some interesting mathematical properties in the analysis of the behavior of its solution (e.g. Layton and Lewandowski [22]). It has been found however, that the model predictions can lead to surprising extensions of the use of the model in technical applications, namely in simulating the turbulent fluid flows. This brief paper should be understood as an introductory note to this novel class of models for applied scientists. It gives the overview of the essential model formulations together with the relevant bibliographical summary.

Keywords: Camassa–Holm equations, turbulent boundary layer

1 Introduction

The Camassa–Holm equations model, was originated in Camassa and Holm [1] and Camassa et al. [2] as a shallow water model with non-linear dispersion. It was later extended in Holm et al. [19] to viscous models for geophysical flows involving stratification and rotation. It is sometimes referred to as the Lagrangian Averaged Navier-Stokes-alpha equations. For a popular introduction into this class of models see e.g. Holm et al. [20].

The governing equations can be written as

\[
\frac{\partial}{\partial t} v + (u \cdot \nabla) v + v_j \nabla u_j = \nu \Delta v - \nabla q + f
\]

\[
\nabla \cdot u = 0
\]

where

\[
v = u - \frac{\partial}{\partial x_i} \left( \alpha^2 \delta_{ij} \frac{\partial}{\partial x_j} \right) u.
\]

Here, \( u \) denotes the averaged physical velocity of the flow, \( q \) is the pressure, \( f \) an external force and \( \nu > 0 \) the viscosity.

The parameter \( \alpha \) represents the typical amplitude of the rapid fluctuations over whose phase the Lagrangian mean is taken in Hamilton’s principle. Physically, alpha is the smallest active length scale participating in the nonlinear interactions, so scales smaller than alpha are swept along by the larger ones and are not allowed to affect their own advection. In the Large-Eddy-Simulation (LES) interpretation, the size alpha can be considered as a natural filter width, which defines the size of a "large" eddy in LES. In other words alpha separates active and passive degrees of freedom (see [6] and [19]).

The whole idea behind the Camassa–Holm equations is that the specific momentum \( v \) is transported by a velocity \( u \) which is smoothed, or filtered, by inverting the elliptic Helmholtz operator \((1 - \alpha^2 \Delta)\), i.e. by taking the \( v = (1 - \alpha^2 \Delta) u \).

In the original versions of the Camassa–Holm model, this parameter \( \alpha \) is assumed to be a constant, due to the assumptions of homogeneity and isotropy of the fluctuations. Near the wall the hypotheses of isotropy and homogeneity in the Lagrangian fluctuations are no more valid. Thus
one of the main problems in using the Camassa-Holm equations in modeling of turbulent flows is related to the finding of an appropriate functional dependence of the length-scale $\alpha$.

The Camassa–Holm equations were introduced in Chen et al. [3], [4], [5] and Foias et al. [12] as a closure model for the Reynolds averaged Navier-Stokes equations in turbulent channel and pipe flows. The model found its application also in turbulent boundary layer flows, jets and wake (see Cheskidov [7], [8], [10]; Mohseni et al. [24] and Putkaradze and Weidman [25]).

For boundary layer turbulence, Cheskidov [7, 8] - see also Cheskidov et al. [10] - obtained an extension of the Prandtl equations for the averaged flow. In the case of a zero pressure gradient flow along a semi-infinite flat plate, namely \{(x, y) : x \geq 0, y = 0\}, a nonlinear fifth-order ordinary differential equation, which is an extension of the Blasius equation, was derived and studied analytically, proving the existence of a two-parameter family of solutions satisfying physical boundary conditions. Matching these parameters with the skin friction coefficient and the Reynolds number based on momentum thickness, an agreement of the solutions with experimental data in the laminar and transitional boundary layers was obtained, as well as in the turbulent boundary layer for moderately large Reynolds numbers.

2 Derivation of the turbulent boundary layer equations

In the following, we briefly present the derivation of the equations for the turbulent boundary layer in the framework of the Camassa–Holm model (see Cheskidov [7, 8]). We consider two-dimensional steady incompressible viscous flow. Let $x$ be the coordinate along the surface, $y$ be the coordinate normal to the surface. Denote also $\mathbf{u} = (u, v)$ to be the velocity of the flow and $\mathbf{v} = (\gamma, \tau)$. We supplement the system (1) with no-slip boundary conditions $\mathbf{u}|_{\partial \Omega} = 0$ as well as

$$\lim_{y \to \infty} \mathbf{u}(x, y) = (u_e(x), 0),$$

for all $x > 0$, where $(u_e(x), 0)$ is the external velocity of the flow outside the boundary layer. We assume that $\alpha$ is a function of $x$ only. More precisely, $\alpha$ will be proportional to the thickness of the boundary layer. If the average velocity $\mathbf{u}$ is stationary in time, (1) becomes

$$u \frac{\partial \gamma}{\partial x} + v \frac{\partial \gamma}{\partial y} + \gamma \frac{\partial u}{\partial x} + \tau \frac{\partial v}{\partial x} = \nu \frac{\partial^2 \gamma}{\partial x^2} + \nu \frac{\partial^2 \gamma}{\partial y^2} - \frac{\partial q}{\partial x},$$

$$u \frac{\partial \tau}{\partial x} + v \frac{\partial \tau}{\partial y} + \gamma \frac{\partial u}{\partial y} + \tau \frac{\partial v}{\partial y} = \nu \frac{\partial^2 \tau}{\partial x^2} + \nu \frac{\partial^2 \tau}{\partial y^2} - \frac{\partial q}{\partial y},$$

$$\frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v = 0$$

Now, fix $l$ on the $x$-axis and define $\epsilon(l)$ to be

$$\epsilon(l) := \frac{1}{\sqrt{R_l}} = \sqrt{\frac{\nu}{u_e l}}.$$

Changing variables

$$x_1 = \frac{x}{l}, \quad y_1 = \frac{y}{\epsilon l}, \quad u_1 = \frac{u}{u_e}, \quad v_1 = \frac{v}{\epsilon u_e}, \quad q_1 = \frac{q}{u_e^2}, \quad \alpha_1 = \frac{\alpha}{\epsilon l}.$$

where $\alpha_1$ does not depend on $y$ variable, and after neglecting terms with high powers of $\epsilon$, we obtain the following turbulent boundary layer equations that generalize Prandtl equations:

$$\nu \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial u}{\partial x} = \nu \frac{\partial^2 w}{\partial y^2} - \frac{\partial q}{\partial x},$$

$$w \frac{\partial \gamma}{\partial y} = - \frac{\partial q}{\partial y},$$

$$\frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v = 0$$
Given any 

where

Then, integrating the second equation in (6)

and substituting in the first equation in (6), we obtain

By finding a solution

In this case,

Now, letting

Substituting the expression (10) for

When \( \alpha = 0 \), the system (9) reduces to the Prandtl equations. Now, making the following assumptions:

where \( \beta \) is a parameter of the boundary layer, which represents the ratio of the averaged size of of

The boundary condition

as

namely

Finally, we have

Finally, we have

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Finally, we have

In this case, \( h \) is a solution to our boundary value problem with \( \beta \) defined as

By finding a solution \( h \) subjected to the condition (13), we find also the value of the parameter \( \beta \).

Since \( h(0) = h'(0) = 0 \), we have to specify three boundary conditions in order to solve (12). The main theoretical result in this direction is given by Cheskidov \[8\]:

**Theorem 2.1** Given any \( a > 0, b \), there exists \( c = c(a, b) \) such that \( h'(\xi) \to \text{const.} \geq 0 \) as \( \xi \to \infty \), where \( h \) is a solution of (12) with \( h(0) = h'(0) = 0 \), \( h''(0) = a \), \( h'''(0) = b \) and \( h''''(0) = c \).
2.1 Comparison with experimental data

Using the non-dimensional distance and velocity
\[ y^+ = \frac{u_* y}{\nu}, \quad u^+ = \frac{u}{u_*} \] (15)

where
\[ u_* = \sqrt{\nu \frac{\partial u}{\partial y} \bigg|_{y=0}} \] (16)
is the friction velocity, we have
\[ u^+ = \left( \frac{R_l}{\sqrt{a \beta}} \right)^{1/4} h' \left( \frac{y^+ \sqrt{\beta}}{\sqrt{a} (R_l)^{1/4}} \right), \] (17)

where \( a = \frac{1}{2} c_f \beta^3 \sqrt{R_l} \), with \( c_f = (u_*/u_e)^2 \) the skin-friction coefficient. Given \( c_f \) and the Reynolds number based on the momentum thickness, namely \( R_\theta \), it is possible to find \( a \), \( b \) and \( R_l \). In a laminar and transitional cases, for given experimental data \( c_f \), \( R_\theta \) and the local Reynolds number \( R_x \), with \( x \) fixed on the horizontal axis, it is possible to take \( R_l = R_x \) and find numerically \( a \) and \( b \). In the turbulent case it is necessary to find \( R_l \) so that the von Kármán logarithmic law holds for the middle inflection point in the logarithmic coordinates. More precisely, assuming that
\[ u^+(y_0^+) = \frac{1}{k} \log (y_0^+) + B, \] (18)

where \( y_0^+ \) is the middle inflection point in the logarithmic coordinates and the constants are the following: \( k \approx 0.4 \) and \( B \approx 5 \); we have to solve the following equation for \( R_l \):
\[ \left( \frac{R_l}{\sqrt{a \beta}} \right)^{1/4} h' \left( \frac{\sqrt{\beta}}{\sqrt{a} (R_l)^{1/4}} \xi_0 \right) = \frac{1}{k} \log \left( \sqrt{a/\beta} \left( \frac{R_l}{\sqrt{a \beta}} \right)^{1/4} \xi_0 \right) + B, \] (19)

where \( \xi_0 \) is the middle inflection point of \( h' (\xi) \) in the logarithmic coordinates.

Varying \( c_f \) and \( R_\theta \), we obtain a family of velocity profiles \( \left\{ u^+_{c_f, R_\theta} \right\} \). Comparison shows that the case \( a + \beta b > 0 \) corresponds to a laminar region of a boundary layer, \( a + \beta b < 0 \) corresponds to a turbulent region of a boundary layer. The case \( a + \beta b = 0 \) corresponds to a transition point.

3 Conclusion

The results of the viscous Camassa-Holm model presented in the above mentioned papers are extremely encouraging as they fit very well both the theoretical profiles and experimental data. A similar approach, but for the Leray–\( \alpha \) model, was successfully used for modeling the transition to turbulence over a rough wall in Cheskidov and Ma [9] with a very promising results.

The class of Lagrangian averaged models (Leray–\( \alpha \), Navier–Stokes–\( \alpha \), etc.) was studied extensively as a possible filtering approach in Large-Eddy-Simulation (LES) e.g. in Geurts and Holm [14, 15] or Ilyin et al. [21]. The comparison with other LES (or DNS) model predictions are shown and discussed in Chen et al. [6], Geurts and Holm [16], Geurts et al. [17], Holm and Titi [18], Foias et al. [11].

As stated in Holm et al. [20], the Lagrangian Averaged Navier–Stokes–\( \alpha \) (LANS–\( \alpha \)) model is the first to use Lagrangian averaging to address the turbulence closure problem by modifying the nonlinearity of the Navier-Stokes equation, instead of its dissipation, thereby providing an alternative way to reach closure without enhancing viscosity. It shows surprisingly good performance in most cases even for the separated boundary layer (Geurts [13]).

The main advantage of this class of models can be seen in the ability to derive them rigorously, in a mathematically and thermodynamically clean way, which is of a great importance in their mathematical analysis as well as in their practical use.

It can be shown, that the classical Navier–Stokes equations (and their solutions) can be obtained as a (singular) limit from the Navier–Stokes–\( \alpha \) model for \( \alpha \to 0 \) (see [12]).
Acknowledgment

The financial support for the present work was provided by the Czech Science Foundation under the grant No. P201-16-03230S and by the Research Plan RVO 67985840.

References


