A POSTERIORI ERROR ESTIMATES AND COUPLING OF ADAPTIVE STRATEGIES FOR A DIFFUSION PROBLEM

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Abstract
In this paper we propose a new hybrid adaptation strategy which preserves the quality of the mesh during refinement and which is coupled to the a posteriori error estimators introduced by M. Vohralik to deal with diffusion problems. To this end, we introduce the New Bisection method restricted to the conformity treatment, thus avoiding the propagation of the refinement as well as the oscillations in the error convergence processes.

Keywords: Finite volumes, finite elements, mesh adaptation, a posteriori error estimator.

1 Introduction
The adaptive method FE-FV (Finite Elements - Finite Volume) is commonly used to approach complex problems arising from several physical phenomena. In many of these applications, adaptive techniques using a posteriori error estimators have become very useful. These estimators measure the quality of the calculated solution and provide information to control the mesh adaptation algorithms. A typical loop of the adaptive method FE-FV by the local refinement is written:

\[ \text{Solve} \rightarrow \text{Estimate} \rightarrow \text{Mark} \rightarrow \text{Refine} \]

2 EF-FV scheme
We consider the following diffusion problem:

\[ -\nabla \cdot a \nabla u = f \quad \text{in } \Omega \]
\[ u = 0 \quad \text{on } \partial \Omega \]

Where \( \Omega \subset \mathbb{R}^2 \), \( f \in L^2(\Omega) \) is a source term, \( a \) represents the permeability tensor. We give on \( \Omega \) a regular triangulation \( T_h \) whose triangles are denoted by \( T \). This triangulation will constitute the primal mesh. We denote \( S_j, 1 \leq j \leq N_S \) the vertices of the primal mesh. \( D_h \) denotes the dual cell, and \( D_h \) the dual mesh. For a triangle \( T \) whose vertices are \( S_1, S_2, \) and \( S_3 \) of the primal mesh, we call \( B \) its barycentre, we call respectively \( \Sigma_{opp}^{1}, \Sigma_{opp}^{2} \) and \( \Sigma_{opp}^{3} \) The edges \([S_2S_3], [S_1S_3]\) and \([S_1S_2] \); \( M_{23}, M_{13} \) and \( M_{12} \) the media of these edges; \( n_{12}, n_{13} \) and \( n_{23} \) outgoing unit normals such that \( n_{pq} \cdot M_{pq} \) and \( n_{pq} \cdot S_{p}S_{q} > 0 \)

2.1 Approximation of the diffusive flux:
The approximation of the diffusive flux is based on an implicit scheme:

\[ -\int_{\partial D_h} a \nabla u \cdot \vec{n} d\sigma = \int_{D_h} f(x)dx \]
\[ -\sum_{T \cap D_h \neq \emptyset} \int_{\partial D_h \cap T_h} a_T \nabla u \cdot \vec{n} d\sigma = \int_{D_h} f(x)dx \]

Where \( a_T \) is an approximation of the permeability tensor on the triangle \( T \). We note the elementary diffusion terms by:

\[ a_{flow}^{12}(T) = |T| a_T \frac{\Sigma_{opp}^{1}}{2|T|} \frac{\Sigma_{opp}^{2}}{2|T|} \frac{n_{opp}^{1}}{n_{opp}^{2}} \]
\[ a_{flow}^{13}(T) = |T| a_T \frac{\Sigma_{opp}^{1}}{2|T|} \frac{\Sigma_{opp}^{3}}{2|T|} \frac{n_{opp}^{1}}{n_{opp}^{3}} \]

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Finally, the finite volume scheme for the flow equation is written:

\[ \sum_{T \in D_h} a_{12}^{\text{flow}}(T)(u_2 - u_1) + a_{13}^{\text{flow}}(T)(u_3 - u_1) = \int_{D_h} f(x)dx \quad (7) \]

3 \ A posteriori error estimates

A posteriori error estimators for complex geometries are based on an approximate post-processing solution locally preserving conservative flows as proposed by Vohralik (see [2]). This class of estimators provides, in particular, a computable upper limit for the numerical error. Before introducing the estimators a posteriori, some definitions and notations are given here. First, \( S_h \) is a (fine) simplicial mesh obtained from the mesh

\[ \text{for all } K \in T_h, \text{RTN}_0^0(K) := [P_0(K)]^d + XP_0(K). \]

A function \( v \in \text{RTN}_0^0(K) \) is written in the following form at a point \( X = (x, y) \in K, v(X) = (a_K + d_K x, b_K + d_K y), d = 2 \) and \( \text{RTN}_0^0(T) := \{ v \in H(\text{div}, \Omega); v|_K \in \text{RTN}_0^0(K), \forall K \in T \}. \) We define \( t_h \in \text{RTN}_0^0(S_h) \) by:

\[ t_h \cdot n_\sigma = - \{ a \nabla u_h \cdot n_\sigma \} \quad (8) \]

with \( \sigma \) is a face of triangulation \( S_h \).
3.1 Estimation of the a posterior error with the energy norm

\[ ||u - u_h|| \leq \left\{ \sum_{D \in D_h} (\eta_{R,D} + \eta_{DF,D})^2 \right\}^{\frac{1}{2}} \]

Where the energy norm is given by:

\[ |a^{\frac{1}{2}} \nabla (u - u_h)| \]

A posteriori error estimator := \( \eta(T_h, D, u_h) \) = \( \left\{ \sum_{D \in D_h} (\eta_{R,D} + \eta_{DF,D})^2 \right\}^{\frac{1}{2}} \)

Residual estimator \( \eta_{R,D} := |a^{\frac{1}{2}} \nabla u_h + a^{-\frac{1}{2}} t_h|_D, \forall D \in D_h \)

Flux estimator \( \eta_{DF,D} := m_{D,a} f - \nabla : t_h|_D, \forall D \in D_h \)

Where: \( m_{D,a}^2 = \frac{C_{P,D} h^2_D}{c_{a,D}} \) if \( D \in D_h^{\text{int}} \), \( m_{D,a}^2 = \frac{C_{F,D} h^2_D}{c_{a,D}} \) if \( D \in D_h^{\text{ext}} \)

Where \( C_{P,D} \) is the constant of the Poincaré inequality and \( C_{F,D} \) the constant of the Friedrichs inequality. \( C_{P,D} \) is equal to \( \frac{1}{2} \) if \( D \) is convex, \( C_{F} \) is equal to 1 on general. The quantities \( C_{P,D} \) and \( C_{F,D} \) are called local error indicators. For these indicators to be useful, they must not overestimate the error locally.

4 Adaptation strategy

4.1 Newest-Vertex-Bisection strategy

\( T: = \) Triangulation of \( \Omega \), for all \( \tau \in T \) we define \( v(\tau) \) the ”newest vertex”.

\( E(\tau): = \) Is the longest edge of \( \tau \), \( v(\tau) \) is the vertex opposite to \( E(\tau) \).

- (R1): The first step consists in dividing the elements by joining \( v(\tau) \) to the middle \( I \) of \( E(\tau) \).
- (R2): \( I \) becomes the ”newest vertex” of each of the two created triangles.
- (R3): Neighbor refinement by RI and conformity (see [1]).

\[ v(\tau) \]

\[ I \quad \text{Conformity} \quad E(\tau) \]

Figure 3: Bisect a triangle and Completion by Newest-Vertex-Bisection strategy

4.2 ADAPT strategy

The principle of the ADAPT method is to start with a coarse mesh which covers the domain of calculation and to evaluate the solution on this mesh. Subsequently, a criterion is chosen which will make it possible to identify the zones where the resolution must be more precise. This method can be repeated in a recursive manner. The triangles marked for refinement are divided into four sub-triangles by joining the midpoints of their edges. Then, for conformity reasons, the refinement levels are adjusted so that the difference in refinement levels between two neighboring cells is at most 1 (see [3]). If the level of refinement of a triangle is equal to one, the triangle is cut in 4 sub triangles using the middle of the edges. If the level of refinement of a triangle is equal to two; Each of the 4 sub triangles will be cut in 4.
4.3 Coupled ADAPT-Newest strategy

Here we propose a new conforming method for the ADPAT method: First we proceed in a first time by refining our mesh by the ADAPT strategy, then for the conformity one uses the method Newest vertex bisection. One notices that there is no more propagation of the refinement on the triangles T1, T2 and T4. The adaptation of the mesh is a good way to obtain a more precise solution with less computation, we must ensure that the evaluation cost of the a posteriori error estimator is much smaller than the cost required to obtain the solution (It should be remembered that usually a global problem must be solved in order to obtain the approximate solution).

5 Numerical results

5.1 Irregular mesh

- Problem:

\[-\text{div} K \nabla p = 0 \quad \text{in } \Omega = (-1, 1)^2\]  \hspace{1cm} (9)

\[p = p_{\text{ex}} \quad \text{on } \partial \Omega\]  \hspace{1cm} (10)

- Heterogeneous permeability

\[K = \begin{cases} 
1.1_2 & \text{if } x \in \Omega_{1,4} \\
5.1_2 & \text{else.} 
\end{cases}\]
- Solution $p \in H^{1+\alpha}(\Omega)$, $\alpha = 0.535$, $a_i, b_i = \text{const.}$

$$p(r, \theta) = r^\alpha(a_i \sin(\alpha \theta) + b_i \cos(\alpha \theta))$$

Figure 6: Refined mesh, exact solution and numerical solution

Figure 7: $L_2$ Error, Energy Error, Estimator and Efficiency

Figure 6 represents the exact and approximate solution in a refined irregular mesh (adaptively) for $\alpha = 0.535$. It is clear that in this problem there is indeed a singularity at the center of the mesh (point of coordinates $(0, 0)$). It is also observed that the mesh is concentrated in the regions where the solution has irregularities. In figure 7, the error between the exact and numerical solution is plotted. Both the estimated error and the energy error, decrease during the refinement of the mesh, which is shown in the following table:

| Nb. elem | $\eta(T_h, D, p_h)$ | $||\nabla (p - p_h)||$ | $||p - p_h||$ | $\frac{2||\nabla (p - p_h)||}{||p - p_h||}$ |
|----------|---------------------|------------------------|--------------|----------------------------------|
| 144      | 0.3847              | 1.999                  | 0.006127     | 4.0994                           |
| 3858     | 1.136               | 0.37519                | 0.002982     | 2.0618                           |
| 11588    | 0.84015             | 0.47503                | 0.0020689    | 1.7686                           |
| 44388    | 0.381               | 0.29893                | 0.0019707    | 1.2745                           |
| 45184    | 0.38066             | 0.29899                | 0.0019701    | 1.2732                           |
| 46053    | 0.38099             | 0.29915                | 0.00196099   | 1.2734                           |
| 46220    | 0.38099             | 0.29919                | 0.00196099   | 1.2734                           |

Table 1: Table of errors and efficiency

This table represents the values of the estimator and the energy error between the exact and the numerical solution as a function of the numbers of the triangles. Here we consider an irregular adaptive mesh and $\alpha = 0.535$. In column 4, we give the $L_2$ error, and finally, in column 5, we give the efficiency index, which is the ratio between the estimated error and the error in the energy norm. We note that this index tends to a value around 1.2, which shows the quality of our estimator despite the fact that the problem has discontinuous coefficients and the exact solution has a singularity at the center.
5.2 Regular mesh

As in the irregular case we give the figures of the regular initial mesh as well as the refined one and the figures of the exact and approximate solution for the same $\alpha = 0.535$. A refinement concentrated in the center is obtained. Even for the regular mesh, the curves of the different errors decrease during the refinement of the mesh and we note that the index of efficiency here also tends towards approximatively 1.2.

![Figure 8: Refined mesh, exact solution and numerical solution](image1)

![Figure 9: L2 error, energy error, estimator and efficiency](image2)

6 Conclusion

Our new hybrid method of mesh adaptation (ADAPT-Newest) has allowed us to avoid the propagation of refinement, to improve the a posteriori error estimators and to have curves of efficiency that converge to values very close to 1. This proves the accuracy of our estimators and the efficiency of our method of adaptation, despite the influence of some constants in the estimators which depends on the convexity of the dual mesh (not always guaranteed) and which are hence unknown in some cases.

References

