INCOMPRESSIBLE AND COMPRESSIBLE VISCOUS FLOW WITH LOW MACH NUMBERS

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Abstract

In this paper we compare incompressible flow and low Mach number compressible viscous flow. Incompressible Navier-Stokes equations were solved by the classical finite element method and compressible Navier-Stokes equations were treated with the aid of discontinuous Galerkin method in space and backward difference method in time. We present numerical results for a flow in a channel which represents a simplified model of the human vocal tract. Presented numerical results give a good correspondence between the the incompressible flow and the compressible flow with low Mach numbers.

Keywords: compressible Navier-Stokes equations, incompressible Navier-Stokes equations, discontinuous Galerkin method, finite element method, low Mach number flow

1 Introduction

In the numerical solution of viscous compressible flow, it is necessary to overcome several issues: resolve accurately shock waves, contact discontinuities and boundary layers. All these phenomena are connected with the simulation of high speed flow with high Mach numbers. However, it appears that the solution of low Mach number flow is also rather difficult. This is caused by the stiff behaviour of numerical schemes and acoustic phenomena appearing in low Mach number flows at incompressible limit. In this case, standard finite volume schemes fail. This led to the development of special finite volume techniques allowing the simulation of compressible flow at incompressible limit, which are based on modifications of the Euler or Navier-Stokes equations. In references [5] and [3] a numerical technique allowing the solution of compressible flow with a wide range of the Mach number without any modification of the governing equations was described. This technique is based on the discontinuous Galerkin finite element method (DGFEM), which combines advantages of the finite volume as well as finite element methods. It employs piecewise polynomial approximations without any requirement on the continuity on interfaces between neighbouring elements.

In this paper we compare incompressible flow and compressible viscous flow for low Mach numbers in a fixed 2D channel, which represents a simplified model of the glottal region of the human vocal tract. Our choice of the domain was motivated by number of works on fluid-structure interaction problems for vocal folds, e.g. [6], [2] and [8].

In Section 2 we formulate the continuous problem of incompressible as well as compressible viscous flow. Section 3 is devoted to the discretization of the problem. For the discretization of the incompressible flow we use standard stabilized finite element method and for the discretization of the compressible flow the discontinuous Galerkin method is applied. Section 4 presents our numerical results concerning the comparison of the incompressible flow and the compressible viscous flow at incompressible limit.
2 Governing equations

2.1 Incompressible flow

The incompressible viscous flow in a computational polygonal domain $\Omega \subset \mathbb{R}^2$ (see Figure 1) and a time interval $[0, T]$, $T > 0$ is governed by the Navier-Stokes equations in the form

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v + \nabla p - \nu \Delta v = 0, \quad \nabla \cdot v = 0,$$

where $v = (v_1, v_2) = v(x, t)$ denotes the velocity vector, $p = p(x, t)$ denotes the kinematic pressure, $\nu$ is the constant kinematic viscosity (i.e. the viscosity divided by the constant fluid density $\rho$).

System (1) is equipped with the initial condition

$$v(x, 0) = v^0(x) \quad x \in \Omega,$$

and with the boundary conditions

a) $v = v_D$ on $\Gamma_I$,  

b) $v = 0$ on $\Gamma_W$,  

c) $-pn - \frac{1}{2}(v \cdot n)v + \nu \frac{\partial v}{\partial n} = 0$ on $\Gamma_O,$

where $n$ denotes the unit outward normal vector to the boundary $\partial \Omega = \Gamma_I \cup \Gamma_O \cup \Gamma_W$, where $\Gamma_I$ denotes the inlet, $\Gamma_O$ the outlet and $\Gamma_W$ the fixed impermeable walls. The boundary condition (3c) is a modification of the so-called ‘do-nothing’ boundary condition, cf. [1]. We set $\lambda^- = \min\{0, \lambda\}$ for $\lambda \in \mathbb{R}$.

![Figure 1](image-url)

Figure 1: The computational domain $\Omega$ with boundary parts: $\Gamma_I$ - inlet, $\Gamma_O$ - outlet, $\Gamma_W$ - fixed impermeable walls.

2.2 Compressible flow

The compressible viscous flow in the domain $\Omega \subset \mathbb{R}^2$ and the time interval $[0, T]$ can be described by the system of the Navier-Stokes equations in the form

$$\frac{\partial w}{\partial t} + \sum_{s=1}^2 \frac{\partial f_s(w)}{\partial x_s} = \sum_{s=1}^2 \frac{\partial R_s(w, \nabla w)}{\partial x_s},$$

where $w$ is the so called state vector defined as $w = (w_1, w_2, w_3, w_4)^T = (\rho, \rho v_1, \rho v_2, E)^T \in \mathbb{R}^4$ and $w^0(x)$. Functions

$$f_s(w) = (\rho v_s, \rho v_1 v_s + \delta_{i,s} p, \rho v_2 v_s + \delta_{2,s} p, (E + p)v_s)^T,$$

$$R_s(w, \nabla w) = \left(0, \tau_{1,s}^{V}, \tau_{2,s}^{V}, \tau_{s,1}^{V}v_1 + \tau_{s,2}^{V}v_2 + \frac{\gamma}{Re Pr} \frac{\partial \theta}{\partial x_s} \right)^T$$

represent the inviscid and viscous terms, respectively. We use the following notation: $\rho$ - density, $v = (v_1, v_2)$ - velocity, $\delta_{i,j}$ - Kronecker symbol, $p$ - pressure, $E$ - total energy, $\theta$ - absolute temperature, $\tau = \{\tau_{ij}\}$ - stress tensor, $\tau^{V} = \{\tau_{ij}^{V}\}$ - viscous part of the stress tensor, $k$ - heat conductivity, $\gamma > 1$ - Poisson adiabatic constant, $Re$ - Reynolds number, $Pr$ - Prandtl number.
System (4) is equipped by the initial condition
\[ w(x, 0) = w^0(x), \quad x \in \Omega, \]
and boundary conditions \( w_B \), see [4], Sections 9.1.2 and 8.3.

3 Discretization

3.1 Incompressible flow

In what follows, by the symbol \( L^2(\Omega) \) the Lebesgue space of square integrable functions on \( \Omega \) is denoted and by \( H^1(\Omega) \) we denote the Sobolev space of square integrable functions together with their first derivatives. By \( H^1(\Omega) \) the space of vector functions with components from \( H^1(\Omega) \) is denoted. Similarly we use notation \( L^2(\Omega) \) for the space of vector functions with components from \( L^2(\Omega) \). The equations (1) are then formulated in a weak sense and the solutions \( v, p \) at any time instant \( t \) are sought in the spaces
\[
W = \{ \varphi \in H^1(\Omega) : \varphi = 0 \text{ on } \Gamma_I \cup \Gamma_W \}
\]
and \( Q = L^2(\Omega), \) respectively. In what follows, by the symbol \( (\cdot, \cdot)_M \) the scalar product in \( L^2(M) \) or \( L^2(M) \) spaces is denoted.

In order to discretize the problem in time, we consider an equidistant partition \( 0 = t_0 < t_1 < \cdots < T, \) \( t_k = k\Delta t \) of the time interval \([0, T] \) with a time step \( \Delta t > 0 \). The time derivative in (1) is then approximated by the second order backward difference formula
\[
\frac{\partial v}{\partial t} \big|_{t=t_{n+1}} \approx \frac{3v^{n+1} - 4v^n + v^{n-1}}{2\Delta t}. \tag{6}
\]
We set \( v^1 := v^0 \).

Now, let us focus on the space discretization at a given time level \( t_{n+1} \).

The spaces \( W \) and \( Q_h \) are approximated by their finite element counterparts \( W_h \subset W \) and \( Q_h \subset Q \). The standard conforming finite element spaces formed by continuous functions are defined with the aid of an admissible triangulation \( T_h \) of \( \Omega \) by
\[
X_h = \left\{ \varphi = (\varphi_1, \varphi_2) \in \left[ C(\overline{\Omega}) \right]^2 : \varphi_i |_{K} \in P_i(K), \forall K \in T_h, i = 1, 2 \right\},
\]
and
\[
Q_h = \left\{ \varphi \in C(\overline{\Omega}) : \varphi_i |_{K} \in P_i(K), \forall K \in T_h \right\}.
\]
The space of test functions is chosen as \( W_h = X_h \cap W \).

For the discrete formulation for any \( U = (v, p) \in X_h \times Q_h, V = (\varphi, q) \in X_h \times Q_h \) we introduce forms
\[
a(U, V) = \left( 3v - v \cdot \nabla v, \varphi \right)_\Omega + (v \nabla v, \nabla \varphi)_\Omega - (p, \nabla \cdot \varphi)_\Omega + (\nabla \cdot v, q)_\Omega,
\]
\[
f(V) = \frac{1}{2\Delta t} (4v^n - v^{n-1}, \varphi)_\Omega.
\]
The stabilization terms containing the SUPG/PSPG stabilization are defined by
\[
\mathcal{L}(U, V) = \sum_{K \in T_h} \delta_K \left( \frac{3v - v \cdot \nabla v}{2\Delta t} - \nu \Delta v + (v \cdot \nabla) v + \nabla p, (v \cdot \nabla) \varphi + \nabla q \right)_K,
\]
\[
\mathcal{F}(V) = \sum_{K \in T_h} \delta_K \left( \frac{4v^n - v^{n-1}}{2\Delta t}, (v \cdot \nabla) \varphi + \nabla q \right)_K,
\]
and the stabilization term for div-div stabilization reads
\[
\mathcal{P}(U, V) = \sum_{K \in T_h} \tau_K (\nabla \cdot v, \nabla \cdot \varphi)_K.
\]
The choice of the parameters $\delta_K = h_K^2$ and $\tau_K = 1$ is carried out according to [7] or [9], where $h_K$ denotes the local element length.

Then the non-linear stabilized discrete problem at a time instant $t = t^{n+1}$ reads: Find $U = (u, p) \in X_h \times Q_h$, $p := p^{n+1}$, $v := v^{n+1}$, such that $v$ satisfies approximately the conditions (3 a-b) and

$$a(U, V) + \mathcal{L}(U, V) + \mathcal{P}(U, V) = f(V) + \mathcal{F}(V),$$

holds for all $V = (\phi, q) \in W_h \times Q_h$. The solution of the arising system of non-linear equations is realized by the Oseen linearization. The linearized problem is solved with the aid of a direct solver.

### 3.2 Compressible flow

For the discretization we employ the discontinuous Galerkin method (DGM). For details we refer to [4]. Let $\mathcal{T}_h$ be a partition of the domain $\Omega$ into finite number of closed triangles $K$. Furthermore, we consider a partition $0 = t_0 < t_1 \ldots < T$ as above with a time step $\Delta t$. Let us denote by $\mathcal{F}_h$ the set of all faces of elements from $\mathcal{T}_h$. It consists of the set of all interior and boundary faces: $\mathcal{F}_h = \mathcal{F}_h^i \cup \mathcal{F}_h^b$. Further let $\mathcal{F}_h^i$, $\mathcal{F}_h^b$ and $\mathcal{F}_h^w$ denote the set of all inlet, outlet and wall faces, respectively. Then $\mathcal{F}_h^b = \mathcal{F}_h^i \cup \mathcal{F}_h^o \cup \mathcal{F}_h^w$. Each face $\Gamma \in \mathcal{F}_h^b$ is associated with a unit normal $\mathbf{n}_\Gamma$, which is an outward normal to $\partial \Omega$ for $\Gamma \in \mathcal{F}_h^b$. Over $\mathcal{T}_h$ we define the broken Sobolev space

$$H^2(\Omega, \mathcal{T}_h) = (H^2(\Omega, \mathcal{T}_h))^d,$$

where $H^2(\Omega, \mathcal{T}_h) = \{ v : \Omega \to \mathbb{R}; v|_K \in H^2(K) \forall K \in \mathcal{T}_h \}$.

The symbols $[u]_\Gamma$ and $(u)_\Gamma$ will denote the jump and the mean value of $u \in H^2(\Omega, \mathcal{T}_h)$ on the face $\Gamma \in \mathcal{F}_h^i$ and $[u]_\Gamma = (u)_\Gamma = u|_{\Gamma}$ for $\Gamma \in \mathcal{F}_h^w$. The approximate solution is sought in the space of piecewise polynomial functions

$$S_{hp} = (S_{hp})^4,$$

where $S_{hp} = \{ v \in L^2(\Omega); v|_K \in P^p(K) \forall K \in \mathcal{T}_h \}$.

In order to define the approximate solution, we introduce the convective form

$$b_h(w, \phi) = \sum_{\Gamma \in \mathcal{F}_h^i} \int_{\Gamma} H(w^{(L)}, w^{(R)}, \mathbf{n}_\Gamma)[\phi] dS + \sum_{\Gamma \in \mathcal{F}_h^w} \int_{\Gamma} H(w^{(L)}, w^{(L)}, \mathbf{n}_\Gamma)\phi dS
- \sum_{K \in \mathcal{T}_h} \sum_{s=1}^2 f_s(w) \frac{\partial \phi}{\partial x_s} dK,$$

where $H$ is a numerical flux. Further, taking into account that

$$R_s(w, \nabla w) = \sum_{k=1}^2 \mathbb{K}_{s,k}(w) \frac{\partial w}{\partial x_k}, \quad s = 1, 2,$$

where $\mathbb{K}_{s,k}$ are $4 \times 4$ matrices depending on $w$, we define the viscous form

$$a_h(w, \phi) = \sum_{K \in \mathcal{T}_h} \int_K \sum_{s,k=1}^2 \left( \mathbb{K}_{s,k}(w) \frac{\partial w}{\partial x_k} \right) \frac{\partial \phi}{\partial x_s} dK
- \sum_{\Gamma \in \mathcal{F}_h^i} \int_{\Gamma} \sum_{s,k=1}^2 \left( \mathbb{K}_{s,k}(w) \frac{\partial w}{\partial x_k} \right) n_s \cdot [\phi] dS
- \sum_{\Gamma \in \mathcal{F}_h^w} \int_{\Gamma} \sum_{s,k=1}^2 \mathbb{K}_{s,k}(w) \frac{\partial w}{\partial x_k} n_s \cdot [w] dS
- \theta \sum_{\Gamma \in \mathcal{F}_h^i} \int_{\Gamma} \sum_{s,k=1}^2 \mathbb{K}_{s,k}(w) \frac{\partial w}{\partial x_k} n_s \cdot (w - w_B) dS,$
where $\theta = 1, 0$ or $-1$ (SIPG, IIPG, resp. NIPG variants). In the scheme we include interior and boundary penalty terms

$$J_h^\alpha(w, \varphi) = \sum_{\Gamma \in F^h} \int_{\Gamma} \sigma[w] : [\varphi] \, dS + \sum_{\Gamma \in F^h} \int_{\Gamma} \sigma(w - w_B) \cdot \varphi \, dS.$$  

The penalty weight $\sigma$ is chosen as $\sigma|_{\Gamma} = \frac{C_W}{\text{diam}(\Gamma)} \frac{Re}{\text{Re}}$ for $\Gamma \in F_h$, where $Re$ is the Reynolds number of the flow and $C_W > 0$ is a suitable constant, which guarantees the stability of the method. Its choice depends on the variant of the used DG method (NIPG, IIPG or SIPG).

Finally we set

$$c_h(w, \varphi) = b_h(w, \varphi) + a_h(w, \varphi) + J_h^\alpha(w, \varphi), \quad w, \varphi \in H^2(\Omega, T),$$

and arrive at the definition of an approximate solution.

**Definition** We say that the functions $w_h$ is the space semidiscrete solution of our problem, if the following conditions are satisfied:

$$w_h \in C^1([0, T]; S_{hp}),$$

$$w_h(0) = \Pi_h w^0,$$

$$\frac{d}{dt}(w_h, \varphi_h) + c_h(w_h, \varphi_h) = 0 \quad \forall \varphi_h \in S_{hp}, \forall t \in (0, T),$$

where $\Pi_h w^0$ is an $S_{hp}$-approximation of $w^0$ from the initial condition.

In the time discretization we use the second-order backward difference formula (BDF) as for the incompressible flow.

**Definition** We say that the finite sequence of functions $w_h^k = w_h(t_k), \quad k = 0, \ldots, r$ is the approximate solution of (1)-(2) computed by the BDF-DGM, if the following conditions are satisfied:

$$w_h^0 = \Pi_h w^0, \quad w_h^{n+1} \in S_{hp}, \quad n = 0, 1, \ldots,$$

$$\frac{3w_h^{n+1} - 4w_h^n + w_h^{n-1}}{2\Delta t} + c_h(w_h^{n+1}, \varphi_h) = 0 \quad \forall \varphi_h \in S_{hp}, \quad n = 1, 2 \ldots$$

We set $w_h^1 = w_h^0$.

### 4 Numerical experiments

In numerical experiments we consider a channel constricted by two bumps of the length 15.4 mm in the glottis region (see Figure 2), which represent a simplified model of vocal folds. The numerical results are obtained for a triangulation consisting of 7790 elements (see Figure 2). We prescribe the following data: the fluid viscosity $\mu = 18 \cdot 10^{-6}$ kg m$^{-1}$ s$^{-1}$, the inlet density $\rho_{in} = 1.225$ kg m$^{-3}$, the outlet pressure $p_{out} = 97611$ Pa, heat conduction coefficient $k = 2.428 \cdot 10^{-2}$ kg m$^{-2}$ K$^{-1}$ and two values of the inlet velocity $v_{in} = 4$ m s$^{-1}$ ($Re = 4356$, Mach number $M = 0.012$) and $v_{in} = 0.4$ m s$^{-1}$ ($Re = 436$, $M = 0.0012$). The Reynolds number was related to the inlet velocity, the inlet density and to the width of the computational domain.

We use time step $\tau = 0.005$ and set $T = 50$. For computing the compressible viscous flow we use the NIPG version of the discontinuous Galerkin method with constant $C_W = 500$ inside of the computational domain and $C_W = 5000$ on the boundary.

We compare the distribution of the pressure and velocity along the axis of symmetry for compressible and incompressible flow at time instants $t = 0.02, 0.08$ and $0.2$ s. Results for $v_{in} = 4$ m s$^{-1}$ are presented in Figures 3-6 and results for $v_{in} = 0.4$ m s$^{-1}$ are shown in Figures 7-10.
Figure 2: Triangulation of the computational domain with sizes of the individual parts in mm.

Figure 3: Distribution of the pressure for $v_{in} = 4 \text{ m s}^{-1}$ along the dimensionless length of the channel (see Figures 1 and 2), on the left the incompressible flow and on the right the compressible flow. The dashed blue diamond marked line stands for time instant $t = 0.02 \text{ s}$, the green circle marked line for $t = 0.08 \text{ s}$ and finally the red triangle marked line for $t = 0.2 \text{ s}$.

Figure 4: Distribution of the magnitude of the velocity for $v_{in} = 4 \text{ m s}^{-1}$ along the dimensionless length of the channel (see Figures 1 and 2), on the left the incompressible flow and on the right the compressible flow. The dashed blue diamond marked line stands for time instant $t = 0.02 \text{ s}$, the green circle marked line for $t = 0.08 \text{ s}$ and finally the red triangle marked line for $t = 0.2 \text{ s}$.
Figure 5: Magnitude of the velocity for $v_{in} = 4 \text{ m s}^{-1}$ in the case of incompressible flow at time instants $t = 0.02, 0.08$ and 0.2 s.

Figure 6: Magnitude of the velocity for $v_{in} = 4 \text{ m s}^{-1}$ in the case of compressible flow at time instants $t = 0.02, 0.08$ and 0.2 s.
Figure 7: Distribution of the pressure for $v_{in} = 0.4 \text{ m s}^{-1}$, on the left the incompressible flow and on the right the compressible flow. The dashed blue diamond marked line stands for time instant $t = 0.02 \text{ s}$, the green circle marked line for $t = 0.08 \text{ s}$ and finally the red triangle marked line for $t = 0.2 \text{ s}$.

Figure 8: Distribution of the magnitude of the velocity for $v_{in} = 0.4 \text{ m s}^{-1}$, on the left the incompressible flow and on the right the compressible flow. The dashed blue diamond marked line stands for time instant $t = 0.02 \text{ s}$, the green circle marked line for $t = 0.08 \text{ s}$ and finally the red triangle marked line for $t = 0.2 \text{ s}$.
Figure 9: Magnitude of the velocity for $v_{in} = 0.4 \text{ m s}^{-1}$ in the case of incompressible flow at time instants $t = 0.02, 0.08$ and $0.2 \text{ s}$.

Figure 10: Magnitude of the velocity for $v_{in} = 0.4 \text{ m s}^{-1}$ in the case of compressible flow at time instants $t = 0.02, 0.08$ and $0.2 \text{ s}$.

From the above figures we see that the pressure drop between the inlet and outlet is in a good correspondence. The same is true for the velocity magnitude in the narrowest part of the channel. Also the vortex structures are similar for the incompressible and compressible flow. Both the velocity and pressure distributions are identical up to the narrowest part of the channel. The differences behind this part are caused by slightly different vortex structures corresponding probably to the transition to turbulence and a different jet declination behind the channel constriction caused by the Coanda effect and interaction of the jet with the large-scale supraglottal eddies.

As a conclusion, it is possible to say that the described numerical method for the solution of the compressible Navier-Stokes equations allows the simulation of very low Mach number flows (e.g. 0.0012) and yields the results comparable with results obtained with the aid of incompressible model.
Our further work will be concentrated to the vocal fold simulation using more realistic geometry with moving and deformed vocal folds solved by incompressible as well as compressible Navier-Stokes systems.

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