Preliminary Study on Correlation Between the Shape Parameter and the Skin Friction Coefficient in Transitional Zero Pressure Gradient Boundary Layer

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Abstract

Paper is concerned with procedures to improve the accuracy of evaluations of the displacement thickness, the momentum thickness and the skin friction from measured mean velocity profiles, namely with regard to transitional boundary layers and to velocity profiles suffering from the lack of points in the immediate vicinity of the wall.

Keywords: transitional boundary layers, integral thicknesses, mean velocity profile

1 Introduction

Experimental investigation of boundary layer development is quite laborious task even if it is focused on the 2D-mean flow field only, i.e. velocity profiles $U(x, y)$ across the flow in the individual sections $x$. It is generally known, that the fundamental features of boundary layer characteristics can be gained without detailed description of the velocity field from the knowledge of the boundary conditions and the integral characteristics in the streamwise direction $x$, namely the wall friction, $\tau_w$, the displacement thickness, $\delta_1$ and the momentum thickness, $\delta_2$ respectively. The credibility of this knowledge depends on the accuracy of the velocity profiles measurements. Here, the accuracy of velocity measurement in individual points $[x, y, z]$ as well as the density of points especially in the immediate vicinity of the the wall $y = 0$ are significant. At least few data of velocity measurements $U_i (i = 1, 2, 3 \ldots)$ in the space $0 < y_i < 0.1 \delta$ are very desirable at the momentum thickness Reynolds number $Re_2 < 10^3$. Then the wall friction can be assumed as proportional to the velocity derivative

$$\tau_w = \mu \left( \frac{\partial U}{\partial y} \right)_{x, y=0}; \quad 0 \leq y < 0.1 \delta$$

(1)

and the value of the derivative can be derived by the linear interpolation $U_i$ versus $y_i$. It is difficult to meet this assumption by means of customary velocity measuring methods in thin boundary layers. According to the author’s experience:

- the CTA measuring method, having developed the well sophistphcated hot-wire wall corrections, meets well the assumption (1) from the boundary layer thickness $\delta$ about 2 ÷ 3 mm, e.g. Jonáš et al [1];

- the measurement by means of a thin flattened Pitot tube (FPT) meets the assumption (1) from $\delta$ about 5 mm, e.g. Jonáš et al [2] if the corrections proposed by MacMillan (see Tropea et.al. [3]) are carefully applied and very short probe shifts, accurately measured, are carried out, e.g. [2] (thickness of probe = 0.18 mm);

- the use of a total-head rake (THR) with simultaneous reading of pressures from individual tubes is useful for preliminary experiments. Such device speeds up the experiments significantly but it hardly gives possibility to meet assumption (1) i.e. direct wall friction measurement. The reasons are following: either the boundary layer thickness is too small or the rake of tubes is too sparse in the immediate vicinity of the wall, $y = 0$. An old THR of thirty Pitot tubes (dia = 0.3 mm) was reconstructed at the IT e.g. Antoš et al [4]. Then some additional measurement and/or extrapolations are necessary otherwise results are not satisfactorily accurate.

The comparison and procedures derived to improve the accuracy of results are presented in this contribution. They are demonstrated on the compounded boundary layer velocity profiles substituting...
accurate measurements with different spacing of points in the given velocity profile. For the sake of simplicity, the zero pressure gradient boundary layers are assumed.

Figure 1: Change of the compounded boundary velocity profiles with the weight parameter of turbulent boundary layer.

2 Introduction of compounded boundary layers

The notion of compounded boundary layer (CBL) is the superposition of two precisely defined flat plate boundary layers (b.l.) alternately occurring in the given section of 2D-mean flow field. The thicknesses $\delta$ and Reynolds number $Re_\delta$ of both b.l. are equal

$$U(\delta) = 0.99 U_L; \quad Re_\delta = \delta U_L/\nu$$

(2)

The first b.l. represents the pure laminar flow with the longitudinal velocity profile given by H. Blasius solution $U_L(x, y)$, e.g. Schlichting [5]. The second b.l. represents fully developed turbulent b.l. with the velocity profile $U_T(x, y)$ compounded from two parts

$$[U_T(x, y)] = y \left( \frac{\partial U}{\partial y} \right)_{y=0} \quad \text{at} \quad y^+ < 6; \quad [U_T(x, y)] = U_L \left( \frac{y}{\delta} \right)^{\nu} \quad \text{at} \quad y^+ \geq 6$$

(3)

The first part, with the velocity proportional to its derivative on the wall ($y = 0$), is valid inside the viscous sublayer and the general Power Law, e.g. [5] describes the remaining part of the layer. It should be mentioned that the wall friction $\tau_w$ cannot be calculated using formulae (1) from velocity profiles $U_T$ determined by the general Power Law. Then the friction $\tau_w$ is calculated in analogy to Wieghardt [6] from the formulae

$$\left( \frac{\tau_w}{\rho U_T^2} \right)_T = \left( C(n) \right)^n \left( \frac{\partial U_L}{\delta} \right)^{\nu}$$

(4)

The function $C(n)$ tabulated in [5] (p.469) was interpolated for the use in this study:

$$C(n) = 1.101 + 1.215 n - 0.0175 n^2$$

(5)

The derivative on the wall is then evaluated from the formulas (1) and (4). The resultant longitudinal mean velocity profile in compounded boundary layer

$$U(x, y) = (1 - c) U_L(x, y) + c U_T(x, y)$$

(6)
depends on the weight parameter \( c \) describing the rate of occurrence of the primary boundary layers resp. the weight of turbulent boundary layer in CBL. Alternating the parameter \( c \), the fundamental features are changing, namely the displacement thickness \( \delta_1 \), the momentum thickness \( \delta_2 \) and the wall friction \( \tau_w \). As an example, the family of CBL-profiles with different values of \( c \) and prescribed joint parameters: b.l. thickness \( \delta = 6 \text{ mm} \), and \( \text{Re}_\delta = 3866 \) are introduced in the Figure 1. To improve the notion on the effect of parameter \( c \) on individual velocity profiles, the dimensionless profiles \((U/U_e + c)\) are plotted in the Figure 1.

The distributions of the shape factor \( H_{12} \) and the momentum thickness Reynolds number \( \text{Re}_2 \) with the weight parameter \( c \) shown in the Figure 2 and the relation between the skin friction coefficient \( C_f \) and the Reynolds number \( \text{Re}_2 \) plotted in the Figure 3 deepen the notion on the compounded boundary layers.

![Figure 2: Distributions of the shape factor and the momentum thickness Reynolds number with the weight parameter \( c \).](image)

![Figure 3: The relation between the skin friction coefficient and the momentum thickness Reynolds number in the compounded boundary layers \((c = 0 \div 1; \delta = 6 \text{ mm}, \text{Re}_\delta = 3866)\).](image)

**3 Evaluation of integral thicknesses and shape factor**

The accurate evaluation of the integral thicknesses \( \delta_1 \) and \( \delta_2 \) and the shape factor \( H_{12} \)

\[
\delta_1 = \int_0^\delta \left(1 - \frac{U}{U_e}\right) dy, \quad \delta_2 = \int_0^\delta \frac{U}{U_e} \left(1 - \frac{U}{U_e}\right) dy, \quad H_{12} = \frac{\delta_1}{\delta_2}
\]  

(7)

is an important step of the boundary layer experimental investigations processing. The determination of boundary layer thickness \( \delta \) can be done exactly and relatively easily. So, the limits of integrals (7) are given. The velocity profile is given by a sequence of points

\[
[y_i, U_i/U_e], \quad y_i > y_i, \quad i = 0, 1, 2, \ldots n
\]  

(8)

The numerical integration is usually performed by using the trapezoidal rule for approximating definite integrals (7). Applying this rule, it follows from the simple judgment, that the difference between the accurate value of the integral and the numerical result depends on the distance between the adjacent points \( y_{i+1} - y_i \) and on the shape (concave/convex) of the integrand \( U = U(x, y_i) \). The shape of the velocity profile is convex (negative second derivative) so the trapezoidal rule underestimates the true value of integrals. Moreover, sometimes the distance from the wall, of the first valid reading \( U(x, y_i) \) is too large in comparison to the boundary layer thickness \( \delta \). This happens namely at measurements by means of a Pitot-tube rake. Then errors in numerical integrals are increasing and a correction is necessary.

A proper correction can be done by exchanging the numerical integration near the wall by the analytical integration of the polynomial interpolation through the first at least three valid readings:

\[
[x, y_0, U_0], [x, y_1, U_1], [x, y_2, U_2], \ldots [x, y_m, U_m] \quad 0, 1, 2, \ldots \leq m
\]  

(9)

Let us assume, the interpolating polynomial

\[
\frac{U}{U_r} = a_0 + a_1 y + a_2 y^2 + a_3 y^3 + a_4 y^4
\]

Taking into account the boundary conditions, \(a_0 = a_2 = 0\) and the number of the unknown coefficients reduces. Then the theoretical regression function for the interpolation has the form

\[
\frac{U}{U_r} = a_1 y + a_3 y^3 + a_4 y^4
\]

This way of improving the accuracy of determining \(\delta_1\) and \(\delta_2\) has been tested on virtual measurements of compounded boundary layers. An example of this procedure, with the CBL introduced in the 2nd section, is presented. The model CBL is minutely “measured” with very short step of the virtual probe (meaningful e.g. a hot wire of CTA)

\[
\left(\frac{y_{n+1} - y_n}{y_n}\right) \sim 0.05
\]

So the velocity profile (8), model profile (M) is determined. Next, the scale of a virtual flattened Pitot probe positions \(y^*_k\) is chosen

\[
\left(\frac{y^*_{k+1} - y^*_k}{y^*_k}\right) \sim 0.1
\]

and the relevant values of velocity \(\left(\frac{U}{U_r}\right)_i\), \(i = 1, 2, 3...\) are assigned to every distance \(y^*\) from the wall. The “Pitot profile” (P) is determined. Finally, the coordinates \(y^*_r\) of the total head Rake individual tubes are chosen

\[
\left(\frac{y^*_{r+1} - y^*_r}{y^*_r}\right) \sim 0.5
\]

and corresponding CBL velocities are calculated. The “Rake profile” (R) is arranged.

Numerical integrations (7) of all profiles M, P and R were performed using the trapezoidal rule

\[
\int_0^\delta \left(\frac{U}{U_r}\right)^m dy = \sum_{j=0}^m \left(\frac{U}{U_r}\right)^m \left(\frac{U}{U_r}\right)^m; \quad y_n = \delta; \quad m = 0, 1, 2
\]

Boundary layer characteristics \(\delta_1\) and \(\delta_2\) and \(H_{12}\) derived from these integrations are marked with the subscripts “MS”, “PS” and “RS” e.g. \(H_{12}^{MS}\).

Furthermore, the integrals (7) of the profiles P and R were divided into two parts, i.e. definite integral of polynomial (11), up to \(y_j = 1.34\) mm and the numerical integration using trapezoidal rule up to the end of the boundary layer \(y_n = \delta\)

\[
\int_0^\delta \left(\frac{U}{U_r}\right)^m dy = \int_0^{y_j} \left(\frac{U}{U_r}\right)^m + \sum_{j=0}^m \left(\frac{U}{U_r}\right)^m \left(\frac{U}{U_r}\right)^m; \quad y_j = 1.3\ mm; \quad y_n = \delta; \quad m = 0, 1, 2
\]
modified integration procedure (16) can satisfactorily reduce the errors (compare characteristics with suffices “MS” and “RP”) in the compounded boundary layers \( (\varepsilon = 0 \div 1; \delta = 6 \text{ mm}, \text{Re}_\delta = 3866) \).

![Figure 4: Distributions of the displacement thickness, the momentum thickness and the shape factor evaluated by the pure trapezoidal rule integration (15) (suffix T) and compounded integral (16) (suffix P).](image)

4 The skin friction estimates in transitional region of boundary layer

In transitional boundary layers, the skin friction \( \tau_w \) can be determined either by a special friction force gauge or from the detailed velocity measurement near the surface. The second method makes possible usage of formula (1), namely if the measurement is done in the region \( y < \delta/10 \).

Looking for another possibility to determine the coefficient \( C_f \) the alternation of “laminar” and “turbulent” periods in boundary layer was remembered. This alternation arises as the consequence of the temporal amplitude modulation of the initial instability wave e.g. Kachanov [7] and Mayle [8]. It was also confirmed by conditional analyses of several experiments e.g. Jonáš et al [9]. Starting from this assumption, the local shape parameter \( H_{12} \) is resolved into the laminar part \( H_L \) and the turbulent part \( H_T \)

\[
H_{12} = gH_L + (1 - g)H_T, \quad 0 \leq g \leq 1
\]

(17)

The value \( H_L = 2.59 \) follows from the Blasius solution (flat plate l. b. l. is supposed). The factor \( H_T \) depends on the Reynolds number \( \text{Re}_\delta \) so it must be determined from the measurement in the region of fully developed turbulent boundary layer. Parameter \( g \) describes the weight of the laminar part.

A similar decomposition of the skin friction coefficient is possible:

\[
C_f = g^*\left(C_f\right)_L + (1 - g^*)\left(C_f\right)_T, \quad 0 \leq g^* \leq 1
\]

(18)

The values \((C_f)_L\) and \((C_f)_T\) are found analogously to the shape parameter above (17). The role of the parameter \( g^* \) is similar to the parameter \( g \) in (17), but there are technically no arguments to suppose numeric equality.

Analyses of ten experiments, done within the framework of ERCOFTAC Test Case T3A+, e.g. [1] and later e.g. Antoš et al [10], revealed the correlation between the shape parameter \( H_{12} = \delta / \bar{\delta}_2 \) and the skin friction coefficient \( C_f \)

\[
g^* = 2.3_{\text{me}} g - 2.1_{43} g^2 + 0.79_{63} g^3, \quad Std (g^*) = 0.02
\]

(19)

The distribution of \( g^* \) versus \( g \) is shown in the Figure 5. The origin specification of experimental points (crosses) is not marked. Only flat plate boundary layers were analysed. The layers were developing under external turbulent flow with the turbulence intensity \( Tu \) ranging from 0.003 to 0.07 and the dissipation length parameter \( Le \) from 2.2 mm up to 33.3 mm.
4 Conclusions

The problem of analysing boundary layer experiment on the basis of boundary layer velocity profiles with low number of measured points in the close vicinity of wall is solved in the paper. It is demonstrated that the accuracy of determination of the displacement and momentum boundary layer thickness can be improved by dividing integrals (7) into two parts (16). The first part is integral of the profile polynomial approximation and the second part is numerical integral using the trapezoidal rule as usual. The procedure was tested on compounded boundary layers (6). The results are shown in the Figure 4.

Investigation of boundary layer transitional region without a proper direct measurement of the wall friction is lacking theoretical support for determination the skin friction coefficient $C_f$. The proposed method provides qualified estimate of $C_f$ from the distribution of the shape factor $H_{12}$. The method is still in progress and it has been validated on smooth flat plate layers at Re2 < 1000.

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References


