DISCUSSION OF EXTREMAL PRINCIPLES FOR TURBULENT FLOW BASED ON HYPOTHESIS OF MAXIMUM ENTROPY PRODUCTION

T. Hyhlík

1 Department of Fluid Dynamics and Thermodynamics, Faculty of Mechanical Engineering, Czech Technical University in Prague, Technická 4, 166 07 Prague 6, Czech Republic

Abstract

The formulation of fundamental thermodynamics inequality representing second law of thermodynamics is given. The entropy production is expressed in the flow of Newtonian fluid. An overall entropy production is evaluated for case of selected basic flow configurations like pipe flow and boundary layer. Both the principle of maximal rate of kinetic energy dissipation and the principle of maximal kinetic energy production are formulated and discussed. By using variational calculus it is shown that flow configurations in equilibrium of turbulent kinetic energy production and dissipation such as channel flow, pipe flow and boundary layer are consistent with principle of maximal entropy production. It is shown that laminar flow is consistent with principle of minimal entropy production.

Keywords: entropy production, extremal principles, turbulent flow.

1 Introduction

Non-negative entropy production is the key property of any thermodynamic system. Modern thermodynamics is using positivity of entropy production to create e.g. the theory of stability [1]. Some authors have noted extremal behaviour of entropy production in the fluid flow, see e.g. works [2, 3]. There is a maximum entropy production principle in statistical thermodynamics [4]. Statistical model of turbulence based on probabilistic definition of mixing entropy exist [4]; mixing entropy is maximized taking into account laws of conservation, boundary and initial conditions and requirement on the probability distribution. Another possibility of using entropy production is previous work of the author [5], where the a posteriori test of turbulence models is created.

2 Entropy production

Material derivative of entropy $S$ in open thermodynamic system is [1]

$$\dot{S} = \int_V \rho s \, dV = -\int_V \frac{q_k}{T} \sigma_k \, dS + \int_V \frac{\bar{q}}{T} \, dV + \int_V \sigma \, dV = -\int_V \frac{q_k}{T} \sigma_k \, dS + \int_V \frac{\bar{q}}{T} \, dV + \dot{S}_{gen},$$

(1)

where $\rho$ is the density, $q_k$ is vector of heat flux, $\bar{q}$ is the density of heat source, $T$ is temperature, $\sigma$ is density of entropy production and $\dot{S}_{gen}$ is overall entropy production. Entropy balance in the material point is

$$\rho \dot{s} = -\frac{1}{T} \frac{\partial q_k}{\partial x_k} + \frac{q_k}{T^2} \frac{\partial T}{\partial x_k} + \frac{\bar{q}}{T} + \sigma.$$  

(2)

Entropy production is expressed by using balance of internal energy [1] as

$$\sigma = \rho \left( \dot{s} - \frac{\dot{u}}{T} \right) - \frac{q_k}{T^2} \frac{\partial T}{\partial x_k} + \frac{1}{T} \sigma_{ki} \frac{\partial v_i}{\partial x_k} \geq 0,$$

(3)

where $\sigma_{ki}$ is stress tensor, $v_i$ is velocity vector and $u$ is internal energy. For the case of fluid is entropy production

$$\sigma = \rho \left( \dot{s} - \frac{\dot{u}}{T} \right) - \frac{q_k}{T^2} \frac{\partial T}{\partial x_k} + \frac{1}{T} (-p \delta_{ki} + \tau_{ki}) \left( S_{ik} + \Omega_{ik} \right) \geq 0,$$

(4)
where $S_{ik}$ is symmetrical part of velocity gradient tensor and $\Omega_{ik}$ is anti-symmetrical part of velocity gradient tensor. It is possible to use symmetry of stress tensor and apply Gibbs definition of entropy, then entropy production is

$$\sigma = \frac{1}{T} \tau_{ij} S_{ij} - \frac{q_k}{T^2} \frac{\partial T}{\partial x_k} \geq 0$$

and overall entropy production is

$$\dot{S}_{gen} = \int_V \sigma dV = \int_V \left( \frac{1}{T} \tau_{ij} S_{ij} - \frac{q_k}{T^2} \frac{\partial T}{\partial x_k} \right) dV \geq 0.$$  

(6)

Density of entropy production in incompressible isothermal fluid flow is

$$\sigma = \frac{2\nu \rho}{T} S_{ij} S_{ij} \geq 0,$$  

(7)

where $\nu$ is kinematic viscosity. It is possible to apply Reynolds averaging [6]

$$\sigma = \frac{2\nu \rho}{T} \left( S_{ij}^\prime S_{ij}^\prime + S_{ij}^\prime \right) = \rho \varepsilon + 2\nu \rho \frac{\partial k}{\partial x_j} \geq 0,$$  

(8)

where $\varepsilon$ is rate of kinetic energy dissipation. The first term is important in the case of turbulent flow and the second plays role in laminar flow and in the turbulent flow close to wall. Kinetic energy $k$ balance equation is according to [6]

$$\rho \frac{d k}{d t} = -\rho u_i^\prime u_j^\prime S_{ij} - \rho \varepsilon + \rho \frac{\partial}{\partial x_j} \left[ \left( \nu + \nu_t \sigma_k \right) \frac{\partial k}{\partial x_j} \right],$$

(9)

where $-\rho u_i^\prime u_j^\prime$ is Reynolds stress tensor, $\nu_t$ is turbulent viscosity and $\sigma_k$ is non-dimensional constant. After substitution we have

$$\sigma = \frac{1}{T} \left\{ -\rho u_i^\prime u_j^\prime S_{ij} + 2\nu \rho S_{ij} S_{ij} - \rho \varepsilon - \rho \frac{\partial}{\partial x_j} \left[ \left( \nu + \nu_t \sigma_k \right) \frac{\partial k}{\partial x_j} \right] \right\} \geq 0.$$  

(10)

### 3 Pipe flow

Entropy balance is in the case of steady adiabatic flow

$$\dot{S}_{gen} = \dot{m} \left( s_{out} - s_{in} \right) \geq 0,$$  

(11)

where $\dot{m}$ is mass flow rate. It is possible to express entropy $s$ as a function of enthalpy $h$ and pressure $p$

$$ds = \frac{1}{T} dh - \frac{1}{\rho T} dp.$$  

(12)

It is known from the balance of energy that $h_{out} = h_{in}$. We can calculate entropy generation rate using pressure integral

$$\dot{S}_{gen} = \dot{m} \int_{s_{out}}^{s_{in}} \left( \frac{1}{\rho T} \right)_{h=const} dp \geq 0.$$  

(13)

In the case of incompressible fluid is

$$\dot{S}_{gen} = \dot{V} \int_{s_{out}}^{s_{in}} dp \geq 0.$$  

(14)

Pressure change is expressed by using friction factor $\lambda$ [7] as

$$dp = -\rho \lambda \frac{dx}{D} \frac{U_{ave}^2}{2},$$  

(15)
where $D$ is pipe diameter and $U_{ave}$ is averaged velocity. Entropy generation rate per meter of the pipe can be expressed by using rate of kinetic energy dissipation

$$\frac{dS_{gen}}{dx} = -\frac{\dot{V}}{T} \frac{dp}{dx} = \frac{\lambda}{2} \frac{U_{ave}^2}{D} = 2\pi \rho \int_0^{D/2} \left[ \varepsilon + \nu \left( \frac{\partial \pi}{\partial r} \right)^2 \right] r dr \geq 0$$

(16)

or by using kinetic energy production

$$\frac{dS_{gen}}{dx} = \frac{\lambda}{2} \frac{U_{ave}^2}{m} = 2\pi \rho \int_0^{D/2} \left[ \varepsilon - u'v' \frac{\partial \pi}{\partial r} + \nu \left( \frac{\partial \pi}{\partial r} \right)^2 \right] r dr \geq 0.$$  

(17)

Pressure change can be expressed by using friction factor as

$$-dp = \frac{\lambda}{D} \frac{dx}{D} \frac{U_{ave}^2}{2} = \frac{2\pi \rho^2 dx}{m} \int_0^{D/2} \left[ -u'v' \frac{\partial \pi}{\partial r} + \nu \left( \frac{\partial \pi}{\partial r} \right)^2 \right] r dr \geq 0.$$  

(18)

It is easy to prove this relation in the case of laminar fully developed pipe flow. Laminar analytical solution is

$$u(r) = \frac{8U_{ave}}{D^2} \left( \frac{D^2}{4} - r^2 \right).$$  

(19)

After substitution is pressure change

$$-dp = \frac{64}{Re} \frac{dx}{D} \frac{U_{ave}^2}{2} \geq 0$$  

(20)

and can be compared with definition (15). Friction factor corresponds to its standard value for laminar flow $\lambda = \frac{64}{Re}$. It has been shown that entropy generation rate is proportional to friction factor $\lambda$ which depends on Reynolds number $Re$ as shown in the figure 1.

### 4 Zero gradient boundary layer

Assume large material control volume whose velocity relative to steady plate is $U_\infty$. The temperature of the fluid in material control volume is $T_\infty$ and pressure is $p_\infty$. The force acting on steady plate is $F_D$. Balance of internal energy $U$ of control volume is then [3]

$$F_D U_\infty = \frac{dU}{dt}.$$  

(21)

Entropy is related to internal energy as

$$dU = T_\infty dS - p_\infty d\left( \frac{1}{\rho} \right).$$  

(22)

Density is constant and entropy production is then

$$\dot{S}_{gen} = \frac{F_D U_\infty}{T_\infty} \geq 0,$$  

(23)

where the force $F_D$ can be expressed as

$$dF_D = c_f \frac{1}{2} \rho U_\infty^2 b dx = u_c^2 \rho b dx,$$  

(24)

where $c_f$ is skin friction coefficient, $u_c$ is friction velocity, $b$ is boundary layer width. Entropy generation per meter of boundary layer is

$$\frac{d\dot{S}_{gen}}{dx} = c_f \frac{1}{2} \rho U_\infty^2 b \frac{\delta}{2T_\infty} = \frac{u_c^2 \rho U_\infty b}{T_\infty} = \frac{\rho b}{T_\infty} \int_0^\delta \left( \varepsilon + 2\nu \delta_{ij} \delta_{ij} \right) dy \geq 0,$$  

(25)
where \( \delta \) is boundary layer thickness. If turbulence is in local equilibrium then we can write entropy production by using kinetic energy production

\[
\frac{d\dot{S}_{\text{gen}}}{dx} = c_f \rho U_\infty^3 b = \frac{u_2^2 \rho U_\infty b}{T_\infty} \int_0^\delta \left( -u_i' u_j' S_{ij} + 2 \nu S_{ij} \right) dy \geq 0. \tag{26}
\]

We can assume that boundary layer is simple shear layer

\[
\frac{d\dot{S}_{\text{gen}}}{dx} = c_f \rho U_\infty^3 b = \frac{u_2^2 \rho U_\infty b}{T_\infty} \int_0^\delta \left[ -u_i' v' \frac{\partial \pi}{\partial y} + \nu \left( \frac{\partial \pi}{\partial y} \right)^2 \right] dy \geq 0. \tag{27}
\]

Skin friction coefficient can be expressed as

\[
c_f = \frac{2}{U_\infty^3} \int_0^\delta \left[ -u_i' v' \frac{\partial \pi}{\partial y} + \nu \left( \frac{\partial \pi}{\partial y} \right)^2 \right] dy \geq 0. \tag{28}
\]

Friction velocity can be expressed as

\[
\frac{\tau_w}{\rho} = u_2^2 = \frac{1}{U_\infty} \int_0^\delta \left[ -u_i' v' \frac{\partial \pi}{\partial y} + \nu \left( \frac{\partial \pi}{\partial y} \right)^2 \right] dy \geq 0 \tag{29}
\]

Let’s try to use only for illustration simple polynomial velocity profile used in the integral method of laminar boundary layer analysis

\[
u(y) = 2\eta - 2\eta^3 + \eta^4, \tag{30}
\]

where \( \eta = y/\delta \).

\[
c_f = \frac{104\nu}{35U_\infty^3 \delta} = \frac{104}{35Re_\delta} \tag{31}
\]

From the Blasius solution [7] we know that \( Re_\delta = 5\sqrt{Re_x} \) and finally we get

\[
c_f = \frac{104}{175\sqrt{Re_x}} = 0.594/\sqrt{Re_x} \tag{32}
\]

This result is relatively close to Blasius solution \( c_f = 0.664/\sqrt{Re_x} \) [7]. It has been shown that entropy generation per meter of boundary layer is proportional to skin friction coefficient \( c_f \) which depends on Reynolds number \( Re_x \) as shown in the figure 1.

### 5 Utilization of variational calculus

This section is dealing with using of variational calculus to try to prove extremal properties of entropy production by using variational calculus.

#### 5.1 Laminar flow

Assume laminar flow of isothermal incompressible fluid where entropy production is

\[
\dot{S}_{\text{gen}} = \frac{2\nu \rho}{T} \int_V S_{ij} \delta S_{ij} dV \geq 0. \tag{33}
\]

If we assume that entropy production has extremal property, then its first variation must be zero

\[
\delta \dot{S}_{\text{gen}} = \frac{4\nu \rho}{T} \int_V S_{ij} \delta S_{ij} dV = 0. \tag{34}
\]
It is possible to recognize the kind of extrema by using second variation
\[
\delta^2 \dot{S}_{gen} = \frac{4 \nu \rho}{T} \int V \delta S_{ij} \delta S_{ij} dV \geq 0. \tag{35}
\]
It is visible that second variation is always positive in the case of the existence of gradient of velocity field and it shows that entropy production is minimized in the case of laminar flow.

5.2 Turbulent flow

It seems impossible to prove extremal properties of turbulent flow in the case of general turbulent flow. An attempt presented here is limited to simple two dimensional shear flow where turbulence is in equilibrium or close to equilibrium. It can be e.g. fully developed channel flow, boundary layer flow etc. In this case the principle of maximal kinetic energy production is applied
\[
\dot{S}_{gen} = \frac{\rho b}{T_{\infty}} \int_0^\delta \left[ \frac{2}{\nu + \nu_t} \frac{\partial \pi}{\partial y} + \nu \left( \frac{\partial \pi}{\partial y} \right)^2 \right] dy = \frac{\rho b}{T_{\infty}} \int_0^\delta \left( \nu + \nu_t \right) \left( \frac{\partial \pi}{\partial y} \right)^2 dy \geq 0, \tag{36}
\]
where eddy viscosity hypothesis is used and \( \nu_t \) is eddy viscosity. First variation of entropy production or in this case first variation of kinetic energy production must be equal to zero in extreme
\[
\delta \left( \frac{\dot{S}_{gen}}{dx} \right) = \frac{\rho b}{T_{\infty}} \int_0^\delta \left[ 2 (\nu + \nu_t) \frac{\partial \pi}{\partial y} \delta \left( \frac{\partial \pi}{\partial y} \right) + \left( \frac{\partial \pi}{\partial y} \right)^2 \delta(\nu_t) \right] dy = 0. \tag{37}
\]
It is necessary to evaluate sign of second variation to recognize the kind of extrema
\[
\delta^2 \left( \frac{\dot{S}_{gen}}{dx} \right) = \frac{\rho b}{T_{\infty}} \int_0^\delta \left[ 2 (\nu + \nu_t) \delta \left( \frac{\partial \pi}{\partial y} \right) \delta \left( \frac{\partial \pi}{\partial y} \right) + 4 \frac{\partial \pi}{\partial y} \delta \left( \frac{\partial \pi}{\partial y} \right) \delta(\nu_t) \right] dy. \tag{38}
\]
If we assume that turbulent flow corresponds to extrema of entropy production or kinetic energy production then we can say that square bracket in the equation for first variation should be zero. The relation between eddy viscosity variation \( \delta(\nu_t) \) and variation of velocity derivation \( \delta \left( \frac{\partial \pi}{\partial y} \right) \) can...
be assumed because eddy viscosity depends on velocity derivative and it is natural to assume the same for their variations. Finally it is possible to evaluate second variation as

$$\delta^2 \left( \frac{\dot{S}_{\text{gen}}}{\text{d}x} \right) = \rho_b \frac{T_\infty}{\delta} \int_0^\delta \left[ -6 \left( \nu + \nu_t \right) \delta \left( \frac{\partial \pi}{\partial y} \right) \delta \left( \frac{\partial \mathcal{M}}{\partial y} \right) \right] \text{d}y \leq 0. \quad (39)$$

It is possible to say that second variation of kinetic energy production is always negative in the case of nonzero velocity derivative. It has been shown that in simple shear flow, where turbulence is in equilibrium or close to it, the production of kinetic energy reaches its maxima. It is possible to say that the principle of maximal kinetic energy production is valid at least in the case of simple shear flow and the same holds for the case of the principle of maximal rate of kinetic energy dissipation.

6 Conclusions

The entropy production is expressed in the case of pipe flow and in the case of zero pressure gradient boundary layer flow. The proposed formulation is tested on the laminar pipe flow and on the laminar zero pressure gradient boundary layer, where correctness of the formulation has been proven. The alternative formulation of friction factor by using entropy production is shown. This formulation can be used as alternative test of turbulence models similarly as in the work [5]. It has been shown that entropy production is proportional to friction factor $\lambda$ or skin friction coefficient $c_f$. Well known distributions in the figure 1 then indicate two possible solutions in certain range of Reynolds number. One solution is corresponding with entropy production minimization and the second is corresponding with maximization of entropy production. Utilization of variational calculus allows to show that laminar flow corresponds with entropy production minimization. The extremal properties of turbulent flow are shown only in the case of simple shear flow, where has been shown that entropy production is maximized. The principle of maximal entropy production is leading to the principle of maximal kinetic energy production in the case of simple shear flow. The utilization of eddy viscosity hypothesis in the prove of extremal properties of entropy production in the turbulent case is the only one weakness. The future goal of the work is to create some method, which can be used in the construction of turbulence models.

Acknowledgement

The support from Czech Science Foundation under Grant No. 14-08888S is gratefully acknowledged.

References