SOME NOTES ABOUT THE HOMOGENIZATION PROBLEM FOR THE NAVIER-STOKES EQUATIONS

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Abstract

We study the homogenization problem for the evolutionary Navier-Stokes system under the critical size of obstacles. Convergence towards the limit system of Brinkman's type is shown under very mild assumptions concerning the shape of the obstacles and their mutual distance.

Keywords: Navier-Stokes system, homogenization, Brinkman's law.

1 Statement of the problem

We consider a bounded spatial domain $\Omega \subset R^3$, together with a family of obstacles (compact sets) $T^1_\varepsilon, \dots, T^{N(\varepsilon)}_\varepsilon$, parameterized by $\varepsilon \to 0$. The motion of an incompressible fluid is governed by the *Navier-Stokes system* of equations

$$\mathrm{div}_x \mathbf{u} = 0 \text{ in } (0,T) \times \Omega_\varepsilon, \tag{1}$$

$$\partial_t \mathbf{u} + \mathrm{div}_x(\mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \mathrm{div}_x \mathbb{S} + \mathbf{f}_\varepsilon \text{ in } (0,T) \times \Omega_\varepsilon, \tag{2}$$

where

$$\Omega_\varepsilon = \Omega \setminus \cup_{i=1}^{N(\varepsilon)} T^i_\varepsilon. \tag{3}$$

The symbol $\mathbf{u}$ denotes the fluid velocity, p is the pressure, $\mathbf{f}_\varepsilon$ denotes a driving force, and $\mathbb{S}$ is the viscous stress tensor given by *Newton's rheological law*

$$\mathbb{S} = \nu(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u}),\ \nu > 0. \tag{4}$$

Problem (1 - 4) is supplemented by the no-slip boundary conditions for the velocity

$$\mathbf{u}|_{\partial\Omega_\varepsilon} = 0, \tag{5}$$

and the initial condition

$$\mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon}. \tag{6}$$

As it is well-known problem (1 - 6) possesses at least one *weak* solution provided $\partial\Omega_\varepsilon$ is sufficiently regular, $\mathbf{f}_\varepsilon \in L^2(0,T;L^2(\Omega_\varepsilon;R^3))$, and $\mathbf{u}_{0,\varepsilon} \in L^2(\Omega_\varepsilon;R^3))$, $\mathrm{div}_x \mathbf{u}_{0,\varepsilon} = 0$, $\mathbf{u}_{0,\varepsilon} \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0$.

2 Conditions on the perforated domains

We consider the so-called critical case, where the diameters of the sets T^i_ε do not exceed the value ε^3, while their mutual distances are larger than ε. More specifically, we assume that

$$T^i_\varepsilon \subset B^i_\varepsilon \equiv \{x \mid |x - x^i_\varepsilon| < r^i_\varepsilon\},\ i = 1, \dots, N(\varepsilon),$$

$$\overline{B}^i_\varepsilon \subset \Omega \text{ for } i = 1, \dots, N(\varepsilon),\ \overline{B}^i_\varepsilon \cap \overline{B}^j_\varepsilon = \emptyset \text{ whenever } i \neq j.$$

Let d^i_ε be a distance between balls B^i_ε, B^j_ε, $j \neq i$, and $\partial\Omega$. Then we suppose the following conditions for the perforation:

$$r^i_\varepsilon < d^i_\varepsilon, \quad \lim_{\varepsilon\to 0} \max_{1\le i\le N(\varepsilon)} d^i_\varepsilon = 0, \tag{7}$$

$$\sum_{i=1}^{N(\varepsilon)} \frac{(r_\varepsilon^i)^2}{(d_\varepsilon^i)^3} \leq C_1 \quad \text{where} \quad C_1 \neq C_1(i,\varepsilon). \tag{8}$$

This distribution of obstacles is called *critical* since for "larger" holes or "shorter" mutual distances the limit velocity would necessarily vanish, while in the opposite case the limit problem would be the same as (1), (2). Note, however, that suitable scaling of the velocities in the former case gives rise to a Darcy-type law as the effective equation (see Allaire [2], [4], Mikelič [6]).

In addition to (7), (8), we assume that the obstacles satisfy the following geometrical condition:

CONDITION (G):

There exists a constant $\omega > 0$ such that at each point $x \in \partial T_\varepsilon^i$ there exists a closed cone C_x with vertex at x and of aperture ω such that

$$C_x \cap T_\varepsilon^i = \{x\}.$$

For a compact set $Q \subset R^3$, we introduce

$$C_{k,l}(Q) = \int_{R^3 \setminus Q} \nabla_x \mathbf{v}^k : \nabla_x \mathbf{v}^l \ \mathrm{d}x, \tag{9}$$

where $\mathbf{v}^k$ is the unique solution of the *model problem*

$$-\Delta_x \mathbf{v}^k + \nabla_x q^k = 0, \ \mathrm{div}_x \mathbf{v}^k = 0 \text{ in } B(x_0,1) \setminus Q, \tag{10}$$

$$\mathbf{v}^k|_{\partial Q} = \mathbf{e}^k, \ \mathbf{v}^k|_{\partial B(x_0,1)} = 0, \tag{11}$$

here $\mathbf{e}^k$, $k = 1,2,3$ is the canonical basis of the space R^3. Let $B(x_0, r)$ be a minimal ball such that $Q \subset B(x_0, r)$, $r \ll 1$. Moreover, let normalize the pressure by the following equality

$$\int_{B(x_0,1)} q^k \, dx = 0.$$

Solution of this problem describes the behavior of the velocity and pressure in the neighbourhood of small set of perforation.

Under the hypotheses (7), (8), it is possible to show that, at least for a suitable subsequence,

$$\lim_{\varepsilon \to 0} \sum_{T_\varepsilon^i \subset G} C_{k,l}(T_\varepsilon^i) = \int_G C_{k,l}(x) \ \mathrm{d}x \tag{12}$$

for any Borel set $G \subset \Omega$, where $\mathbb{C} = \{C_{k,l}\}_{k,l=1}^3$, $\mathbb{C} \in L^\infty(\Omega; R^{3\times 3}_{\mathrm{sym}})$. It can be shown that the matrix $\mathbb{C}$ is constant in the case of periodically distributed obstacles of identical (rescaled) shape, see Allaire [3].

3 Definition of a weak solution

We say that $\mathbf{u}_\varepsilon$ is a *weak solution* of problem (1 - 6) if

- $\mathbf{u}_\varepsilon$ belongs to the class $L^\infty(0,T; L^2(\Omega_\varepsilon; R^3)) \cap L^2(0,T; W^{1,2}(\Omega_\varepsilon; R^3))$;
- $\mathrm{div}_x \mathbf{u}_\varepsilon = 0$ a.a. in $(0,T) \times \Omega_\varepsilon$;
- the integral identity

$$\int_0^T \int_{\Omega_\varepsilon} \Big(\mathbf{u}_\varepsilon \cdot \partial_t \mathbf{w} + (\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \nabla_x \mathbf{w} \Big) \mathrm{d}x \ \mathrm{d}t = - \int_{\Omega_\varepsilon} \mathbf{u}_{0,\varepsilon} \cdot \mathbf{w}(0,\cdot) \ \mathrm{d}x \tag{13}$$

$$+ \int_0^T \int_{\Omega_\varepsilon} \mathbb{S} : \nabla_x \mathbf{w} \ \mathrm{d}x \ \mathrm{d}t - \int_0^T \int_{\Omega_\varepsilon} \mathbf{f}_\varepsilon \cdot \mathbf{w} \ \mathrm{d}x \ \mathrm{d}t$$

  holds for any test function $\mathbf{w} \in C_c^\infty([0,T) \times \Omega_\varepsilon; R^3)$, $\mathrm{div}_x \mathbf{w} = 0$;

- the energy inequality

$$\int_{\Omega_\varepsilon} \frac{1}{2}|\mathbf{u}_\varepsilon|^2(\tau,\cdot)\ \mathrm{d}x + \nu \int_0^\tau \int_{\Omega_\varepsilon} |\nabla_x \mathbf{u}_\varepsilon|^2\ \mathrm{d}x\ \mathrm{d}t$$
$$\leq \int_{\Omega_\varepsilon} \frac{1}{2}|\mathbf{u}_{0,\varepsilon}|^2\ \mathrm{d}x + \int_0^\tau \int_{\Omega_\varepsilon} \mathbf{f}_\varepsilon \cdot \mathbf{u}_\varepsilon\ \mathrm{d}x\ \mathrm{d}t$$

  holds for a.a. $\tau > 0$.

Extending $\mathbf{u}$ to be zero outside Ω_ε, for

$$\{\mathbf{u}_{0,\varepsilon}\}_{\varepsilon>0} \text{ bounded in } L^2(\Omega; R^3), \tag{14}$$

$$\{\mathbf{f}_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^2(0,T; W^{-1,2}(\Omega; R^3)), \tag{15}$$

the associated family of weak solutions $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ of problem (1 - 6) satisfies

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ weakly-}(*) \text{ in } L^\infty(0,T; L^2(\Omega; R^3)) \text{ and weakly in } L^2(0,T; W_0^{1,2}(\Omega; R^3)), \tag{16}$$

at least for a suitable subsequence.

It is easy to check that the limit $\mathbf{u}$ satisfies the incompressibility constraint (1) a.a. in $(0,T)\times\Omega$, however, performing the passage in the momentum equation (2) is more delicate. The collective effect of friction forces imposed on the fluid by each obstacle results, in general, in a new term of a *Brinkman* type appearing in the limit problem. The effective momentum equation satisfied by the limit velocity field $\mathbf{u}$ reads

$$\partial_t \mathbf{u} + \mathrm{div}_x(\mathbf{u}\otimes\mathbf{u}) + \mathbb{C}\mathbf{u} + \nabla_x p = \mathrm{div}_x \mathbb{S} + \mathbf{f}, \tag{17}$$

where $\mathbf{f}$ is a weak limit of the sequence $\{\mathbf{f}_\varepsilon\}_{\varepsilon>0}$.

In order to justify the limit passage from (2) to (17), it is necessary to control the pressure in the associated stationary Stokes system

$$-\Delta \mathbf{v} + \nabla_x q = \mathbf{f}_\varepsilon \text{ in } \Omega_\varepsilon,\ \mathbf{v}|_{\partial\Omega_\varepsilon} = 0. \tag{18}$$

As observed in the seminal work of Tartar [7], this step requires the existence of *restriction operator* $\mathcal{R}_\varepsilon$ enjoying the following properties:

- $\mathcal{R}_\varepsilon : W_0^{1,2}(\Omega; R^3) \to W_0^{1,2}(\Omega_\varepsilon; R^3)$ is a bounded linear operator,

$$\|\mathcal{R}_\varepsilon[\mathbf{v}]\|_{W_0^{1,2}(\Omega_\varepsilon;R^3)} \leq c\|\mathbf{v}\|_{W_0^{1,2}(\Omega;R^3)}, \tag{19}$$

  with c independent of ε.

- $$\mathcal{R}_\varepsilon[\mathbf{v}] = \mathbf{v} \text{ for any } \mathbf{v} \in W_0^{1,2}(\Omega_\varepsilon; R^3). \tag{20}$$

- $$\mathrm{div}_x \mathcal{R}_\varepsilon[\mathbf{v}] = 0 \text{ whenever } \mathrm{div}_x \mathbf{v} = 0. \tag{21}$$

The construction of the restriction operator $\mathcal{R}_\varepsilon$ based on the recent results of Acosta et al. [1], Diening et al. [5]. It can be constructed under very mild restrictions imposed on the shape of the obstacles $\{T_\varepsilon^i\}_{i=1,\varepsilon>0}^{N(\varepsilon)}$, in particular if all of them are convex.

4 Main result

Our main result reads as follows.

Theorem Let $\{\Omega_\varepsilon\}_{\varepsilon>0} \subset R^3$ be a family of domains given by (3), where T^i_ε, $i = 1, \dots, N(\varepsilon)$, satisfy (7), (8), together with condition (G). Assume that

$$\left\{ \begin{array}{c} \mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \text{ weakly in } L^2(\Omega; R^3), \\ \\ \mathbf{f}_\varepsilon \to \mathbf{f} \text{ weakly in } L^2((0,T)\times\Omega; R^3). \end{array} \right\}$$

Let $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ be a family of weak solutions of problem (1 - 6).

Then, at least for a suitable subsequence,

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ in } L^2((0,T)\times\Omega; R^3)) \text{ and weakly in } L^2(0,T; W^{1,2}(\Omega; R^3)), \tag{22}$$

where $\mathbf{u}$ is a weak solution of the problem

$$\partial_t \mathbf{u} + \mathrm{div}_x(\mathbf{u}\otimes\mathbf{u}) + \mathbb{C}\mathbf{u} + \nabla_x p = \mathrm{div}_x \mathbb{S} + \mathbf{f} \text{ in } (0,T)\times\Omega, \tag{23}$$

$$\mathrm{div}_x \mathbf{u} = 0 \text{ a.a. in } (0,T)\times\Omega, \tag{24}$$

with $\mathbb{C}$ given by (12), supplemented with the initial condition

$$\mathbf{u}(0,\cdot) = \mathbf{u}_0, \tag{25}$$

and the boundary condition

$$\mathbf{u}|_{\partial\Omega} = 0. \tag{26}$$

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