NON-NEWTONIAN FLUIDS WITH NONSTANDARD RHEOLOGY - EXISTENCE OF SOLUTIONS

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Abstract

Our aim is to present a few results concerning existence of weak solutions to unsteady flow of incompressible non-Newtonian fluids. In particular, we are interested in dynamics of non-Newtonian fluids of a nonstandard rheology, more general then of power-law type. In considered problems the nonlinear highest order term – stress tensor – is monotone and its behaviour – coercivity/growth condition – is given with help of some general convex function. In our research we cover both cases: shear thickening and shear thinning fluids and as well as anisotropic and non-homogenous behaviour of the stress tensor. Such a formulation requires a general framework for the function space setting, therefore we work with non-reflexive and non-separable anisotropic Orlicz and Musielak-Orlicz spaces.

Keywords: incompressible non-Newtonian fluids, weak solutions, Orlicz spaces, Musielak-Orlicz spaces

1 Introduction

We would like to recall here some of our existing results to problems capturing flows of non-Newtonian fluids with non-standard rheology. In particular, we want to present some studies which allow to consider the phenomena of viscosity changing significantly under various stimuli like shear rate, magnetic or electric field described with help of some general convex function. Therefore we may investigate materials whose properties can be described not only by the dependence on constant viscosity. Our research concerns existence and properties of solutions to systems of equations coming from fluid mechanics. We concentrate on the case of an incompressible fluid for which equations can take the following general form

$$\begin{aligned}
\partial_t \varrho + \operatorname{div}_x(\varrho \boldsymbol{u}) &= 0 \quad \text{in} \quad Q,\\
\partial_t(\varrho \boldsymbol{u}) + \operatorname{div}_x(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) - \operatorname{div}_x \mathbf{S}(t, x, \varrho, \mathbf{D}\boldsymbol{u}) + \nabla_x p &= \varrho \boldsymbol{f} \quad \text{in} \quad Q,\\
\operatorname{div}_x \boldsymbol{u} &= 0 \quad \text{in} \quad Q,\\
\boldsymbol{u}(0, x) &= \boldsymbol{u}_0 \quad \text{in} \quad \Omega,\\
\varrho(0, x) &= \varrho_0 \quad \text{in} \quad \Omega,\\
\boldsymbol{u}(t, x) &= 0 \quad \text{on} \quad (0, T) \times \partial\Omega,
\end{aligned} \tag{1}$$

where $\varrho : Q \to \mathbb{R}$ is the mass density, $\boldsymbol{u} : Q \to \mathbb{R}^3$ denotes the velocity field, $p : Q \to \mathbb{R}$ the pressure, $\mathbf{S}$ the stress tensor, $\boldsymbol{f} : Q \to \mathbb{R}^3$ given outer sources. The set $\Omega \subset \mathbb{R}^3$ is a bounded domain with a regular boundary $\partial\Omega$ (of class, say $C^{2+\nu}$, $\nu > 0$, to avoid unnecessary technicalities connected with smoothness). We denote by $Q = (0, T) \times \Omega$ the time-space cylinder with some given $T \in (0, +\infty)$. The tensor $\mathbf{D}\boldsymbol{u} = \frac{1}{2}(\nabla_x \boldsymbol{u} + \nabla_x^T \boldsymbol{u})$ is a symmetric part of the velocity gradient.

In order to close the system we have to state the constitutive relation, rheology, which describes the relation between $\mathbf{S}$ and $\mathbf{D}\boldsymbol{u}$. In our considerations we do not want to assume that $\mathbf{S}$ has only polynomial-structure, i.e. $\mathbf{S} \approx (\kappa + |\mathbf{D}\boldsymbol{u}|)^{p-2}\mathbf{D}\boldsymbol{u}$ or $\mathbf{S} \approx (\kappa + |\mathbf{D}\boldsymbol{u}|^2)^{(p-2)/2}\mathbf{D}\boldsymbol{u}$ (where $\kappa > 0$). Standard growth conditions of the stress tensor, namely polynomial growth, see e.g. [11, 37]

$$\begin{aligned}
|\mathbf{S}(\mathbf{D}\boldsymbol{u})| &\leqslant c(1 + |\mathbf{D}\boldsymbol{u}|^2)^{(p-2)/2}|\mathbf{D}\boldsymbol{u}|\\
\mathbf{S}(\mathbf{D}\boldsymbol{u}) : \mathbf{D}\boldsymbol{u} &\geqslant c(1 + |\mathbf{D}\boldsymbol{u}|^2)^{(p-2)/2}|\mathbf{D}\boldsymbol{u}|^2
\end{aligned} \tag{2}$$

can not suffice to describe nonstandard behaviour of the fluid. Motivated by the significant shear thickening phenomenon we want to investigate the processes where the growth is faster than polynomial and possibly different in various directions of the shear stress. Also the case of growth close to linear can be covered in this way. A viscosity of the fluid is not assumed to be constant, can depend on density and full symmetric part of the velocity gradient as well as can be inhomogeneous in space (i.e. depend on the point in considered domain). Therefore we formulate the growth conditions of the stress tensor using a general convex function M called an $\mathcal{N}-$function (the definitions of an $\mathcal{N}-$function M and its complementary function M^* appear in Section 4.1) similarly as in [19, 21, 22, 24, 25, 52, 53, 54, 55, 56]:

$$\mathbf{S}(x, \mathbf{D}\boldsymbol{u}) : \mathbf{D}\boldsymbol{u} \geqslant c\{M(x, \mathbf{D}\boldsymbol{u}) + M^*(x, \mathbf{S}(x, \mathbf{D}\boldsymbol{u}))\}\,. \tag{3}$$

Now we are able to describe the effect of rapidly shear thickening and shear thinning fluids as well as its anisotropic and/or inhomogeneous behaviour.

The appropriate spaces to capture such formulated problem are generalized Orlicz spaces, often called Orlicz-Musielak spaces. (In classical case, i.e. with polynomial growth conditions, the proper space setting is standard Lebesgue and Sobolev spaces.) We also allow the stress tensor to depend on x, this provides the possibility to consider electro- and magnetorheological fluids and significant influence of magnetic and magnetic field on the increase of viscosity. Thus we use the generalized Orlicz spaces, often called Orlicz-Musielak spaces (see [38] for more details). For definitions and preliminaries of $\mathcal{N}-$functions and Orlicz spaces see Section 4.1. Contrary to [38] we consider the $\mathcal{N}-$function M not dependent only on $|\boldsymbol{\xi}|$, but on whole tensor $\boldsymbol{\xi}$. It results from the fact that the viscosity may differ in different directions of symmetric part of velocity gradient $\mathbf{D}\boldsymbol{u}$. Hence we want to consider the growth condition for the stress tensor dependent on the whole tensor $\mathbf{D}\boldsymbol{u}$, not only on $|\mathbf{D}\boldsymbol{u}|$. The spaces with an $\mathcal{N}$–function dependent on vector-valued argument were investigated in [48, 49, 50].

In our considerations condition (3) forces us to use Orlicz, Orlicz-Sobolev spaces, defined by the $\mathcal{N}$–function. We want to emphasise that we do not want to assume that M satisfies the so-called Δ_2–condition. Therefore we lose a wide range of facilitating properties of function spaces that one normally works with. Namely, if M does not satisfy the Δ_2–condition then our spaces are not reflexive, separable, smooth functions are not dense with respect to the norm. The lack of such assumption is a reason of many delicate and deep handicaps. Therefore we need to obtain the result using more sophisticated methods than in the classical case. Our investigations are directed to existence and properties of solutions.

Let us emphasised that one of the main problems in our considerations is that the $\triangle_2$–condition can not be satisfied and we lose many facilitating properties. An interesting obstacle here is the lack of the classical integration by parts formula, cf. [17, Section 4.1]. To extend it for the case of generalized Orlicz spaces we would essentially need that $C^\infty-$functions are dense in $L_M(Q)$ and $L_M(Q) = L_M(0, T; L_M(\Omega))$. The first one only holds if M satisfies the Δ_2–condition. The second one is not the case in Orlicz and generalized Orlicz spaces. We recall the proposition from [5] (although it is stated for Orlicz spaces with $M = M(|\boldsymbol{\xi}|)$).

Proposition 1.1 *Let I be the time interval, $\Omega \subset \mathbb{R}^d$, $M = M(|\boldsymbol{\xi}|)$ an $\mathcal{N}$–function, $L_M(I \times \Omega)$, $L_M(I; L_M(\Omega))$ the Orlicz spaces on $I \times \Omega$ and the vector valued Orlicz space on I respectively. Then $L_M(I \times \Omega) = L_M(I; L_M(\Omega))$, if and only if there exist constants k_0, k_1 such that*

$$k_0 M^{-1}(s) M^{-1}(r) \leqslant M^{-1}(sr) \leqslant k_1 M^{-1}(s) M^{-1}(r) \tag{4}$$

for every $s \geqslant 1/|I|$ and $r \geqslant 1/|\Omega|$.

One can conclude that (4) means that M must be equivalent to some power p, $1 < p < \infty$. Hence, if (4) should hold, very strong assumptions must be satisfied by M. Surely they would provide $L_M(Q)$ to be separable and reflexive.

Substantial part of our studies is motivated by a significant shear thickening phenomenon. Therefore we want to investigate the processes where the growth of the viscous stress tensor is faster than polynomial. Hence $\mathcal{N}$–function defining a space does not satisfy the Δ_2–condition. At the beginning our attention is directed to incompressible fluids with non-constant density, see section 5 and [54] by Wróblewska-Kamińska. We include the case of different growth of the stress tensor in various directions of the shear stress and possible dependence on some outer field. The second problem concerns the motion of rigid bodies in shear thickening fluid, see [56]. The bodies have a nonhomogeneous structure and are immersed in a homogenous incompressible fluid.

Omitting in this case the assumption of Δ_2–conditions has physical motivations. The requirement for avoiding collisions is a high enough integrability of the shear stress (at least in L^4). Hence it is natural to consider an $\mathcal{N}$–function of high growth e.g. exponential.

The presence of convective term in problems mentioned above allowed us to consider only shear thickening fluids. It is a consequence of the fact that the convective term enforced the restriction for the growth of an $\mathcal{N}$-function, namely $M(\cdot) \geqslant c|\cdot|^q$ for some exponent $q \geqslant \frac{3d+2}{d+2}$. If we assume that the flow is slow, then it is reasonable to neglect the convective term and we are able to skip the assumption on the lower growth of M (and consequently the bound for M^*). It opens a possibility to include flows of different behaviour. In particular the growth of the viscous stress tensor can be close to linear and is prescribed by an anisotropic $\mathcal{N}$–function whose complementary does not have to satisfy the Δ_2–condition and therefore we are able also to investigate the flow of shear thinning fluid.

In the following paper we want to give the reader better insight into the above results and we present here short overview of the considered problems.

2 Non-Newtonian fluid and motivations

Our interest is directed to the phenomena of viscosity increase under various stimuli: shear rate, magnetic or electric field. Particularly we want to focus on shear thickening (STF) and magnetorheological (MR) fluids. Both types of fluid have the ability of transferring rapidly from liquid to solid-like state and this phenomenon is completely reversible, and the time scale for the transmission is of the order of a millisecond. The magnetorheological fluids [57] found their application in modern suspension system, clutches or crash-protection systems in cars and shock absorbers providing seismic protection.

In particular we are interested in fluids having viscosity which increases dramatically with increasing shear rate or applied stress, i.e. we want to consider shear thickening fluids, which can behaves like a solid when it encounters mechanical stress or shear. STF moves like a liquid until an object strikes or agitates it forcefully. Then, it hardens in a few milliseconds. This is the opposite of a shear-thinning fluid, like paint, which becomes thinner when it is agitated or shaken. The fluid is a colloid, consists of solid particles dispersed in a liquid (e.g. silica particles suspended in polyethylene glycol). The particles repel each other slightly, so they float easily throughout the liquid without clumping together or settling to the bottom. But the energy of a sudden impact overwhelms the repulsive forces between the particles – they stick together, forming masses called hydroclusters. When the energy from the impact dissipates, the particles begin to repel one another again. The hydroclusters fall apart, and the apparently solid substance reverts to a liquid.

Possible application for fluids with changeable viscosity appears in military armour. The so-called STF-fabric produced by simple impregnation process of e.g. Kevlar makes it applicable to any high-performance fabric. The resulting material is thin and flexible, and provides protection against the risk of needle, knife or bullet contact that face police officers and medical personnel [6, 29, 34].

One of the example is a magnetorheological fluid, which consists of the magnetic particles suspended within the carrier oil distributed randomly in suspension under normal circumstances. When a magnetic field is applied, the microscopic particles align themselves along the lines of magnetic flux. In the fluid contained between two poles, the resulting chains of particles restrict the movement of the fluid, perpendicular to the direction of flux, effectively increasing its viscosity. Consequently mechanical properties of the fluid may be anisotropic.

On the other hand we can consider the constitutive relation for fluids with dependence on outer field, in particular, we mean electrorheological fluids. In this case, from representation theorem it follows the stress tensor may possesses growth of different powers in various directions of $\mathbf{D}$ (cf. [45, 54]). In such situation the mechanical minimal assumptions are satisfied and we can not exclude constitutive relation of anisotropic behaviour.

We can observe that the case of stress tensors having convex potentials (additionally vanishing at $\mathbf{0}$ and symmetric w.r.t. the origin) significantly simplifies verifying condition (3). For finding $\mathcal{N}-$functions M and M^* we take an advantage of the following relation

$$M(\boldsymbol{\xi}) + M^*(\nabla M(\boldsymbol{\xi})) = \boldsymbol{\xi} : \nabla M(\boldsymbol{\xi}) \tag{5}$$

holding for all $\boldsymbol{\xi} \in \mathbb{R}^{3\times3}_{\mathrm{sym}}$, cf. [44]. This corresponds to the case when the Fenchel-Young inequality for $\mathcal{N}$–functions becomes an equality. Once we have a given function $\mathbf{S}$, for simplicity consider it in the form $\mathbf{S}(\mathbf{D}\boldsymbol{u}) = 2\mu(|\mathbf{D}\boldsymbol{u}|^2)\mathbf{D}\boldsymbol{u}$, then choosing $M(x,\boldsymbol{\xi}) = M(\boldsymbol{\xi}) = \int_0^{|\boldsymbol{\xi}|^2} \mu(\boldsymbol{\alpha})\, d\boldsymbol{\alpha}$ provides that (3) is satisfied with a constant $c = 1$. For such chosen M we only need to verify whether the $\mathcal{N}-$function–conditions, i.e, behaviour in/near zero and near infinity, are satisfied. The monotonicity of $\mathbf{S}$ follows from the convexity of the potential.

Our assumptions can capture shear dependent viscosity function which includes power-law and Carreau-type models which are quite popular among rheologists, in chemical engineering, and colloidal mechanics (see [36] for more references). Nevertheless we want to investigate also more general constitutive relations like non-polynomial growth $\mathbf{S} \approx |\mathbf{D}\boldsymbol{u}|^p \ln(1 + |\mathbf{D}\boldsymbol{u}|)$ or of anisotropic behaviour e.g. $\mathbf{S}_{i,j} \approx |\cdot|^{p_{ij}}[\mathbf{D}\boldsymbol{u}]_{i,j}$, $i,\ j = 1, 2, 3$.

Our particular interest is directed here to the rheology close to linear in at least one direction (see section 6 and [22]). We do not assume that the $\mathcal{N}-$ function satisfies the Δ_2-condition in case of star-shaped domains. For other domains we need to assume some conditions on the upper growth of M, however this does not contradict with a goal of describing the rheology close to linear. There is a wide range of fluid dynamics models obeying these conditions, we mention here two constitutive relations: Prandtl-Eyring model, cf. [9], where the stress tensor $\mathbf{S}$ is given by

$$\mathbf{S} = \eta_0 \frac{\operatorname{ar\,sinh}(\lambda|\mathbf{D}\mathrm{u}|)}{\lambda|\mathbf{D}\boldsymbol{u}|}\mathbf{D}\boldsymbol{u}$$

and modified Powell-Eyring model cf. [41]

$$\mathbf{S} = \eta_\infty \mathbf{D}\boldsymbol{u} + (\eta_0 - \eta_\infty)\frac{\ln(1 + \lambda|\mathbf{D}\boldsymbol{u}|)}{(\lambda|\mathbf{D}\boldsymbol{u}|)^m}\mathbf{D}\boldsymbol{u}$$

where η_∞, η_0, λ, m are material constants. Our attention in the present section is particularly directed to the case $\eta_\infty = 0$ and $m = 1$.

Both models are broadly used in geophysics, engineering and medical applications, e.g. for modelling of glacier ice, cf. [31], blood flow, cf. [42, 43] and many others, cf. [2, 40, 47].

In section 6 and in [22] our considerations concern the simplified system of equations of conservation of mass and momentum. Indeed, the convective term $\operatorname{div}_x(\boldsymbol{u} \otimes \boldsymbol{u})$ is not present in the equations. The motivation for considering such a simplified model is twofold. If the flow is assumed to be slow, then the inertial term $\operatorname{div}_x(\boldsymbol{u} \otimes \boldsymbol{u})$ can be assumed to be very small and therefore neglected, hence the whole system reduces to a generalized Stokes system. Another situation is the case of simple flows, e.g. Poisseuille type flow, between two fixed parallel plates, which is driven by a constant pressure gradient (see [30]). With regards to blood flows the importance of considering simple flows arises since the geometry of vessels can be simplified to a flow in a pipe. The analysis of both models in steady case (also without convective term) through variational approach was undertaken by Fuchs and Seregin in [15, 16].

3 State of art

The mathematical analysis of time dependent flow of homogeneous non-Newtonian fluids with standard polynomial growth conditions was initiated by Ladyzhenskaya [32, 33] where the global existence of weak solutions for $p \geqslant 1 + (2d)/(d + 2)$ was proved for Dirichlet boundary conditions. Later the steady flow was considered by Frehse at al. in [12], where the existence of weak solutions was established for the constant exponent $p > \frac{2d}{d+2}$, $d \geqslant 2$ by Lipschitz truncation methods.

Wolf in [51] proved existence of weak solutions to unsteady motion of an incompressible fluid with shear rate dependent viscosity for $p > 2(d + 1)/(d + 2)$ without assumptions on the shape and size of Ω employing an L^∞–test function and local pressure method. Finally, the existence of global weak solutions with Dirichlet boundary conditions for $p > (2d)/(d + 2)$ was achieved in [4] by Lipschitz truncation and local pressure methods.

Most of the available results concerning nonhomogeneous incompressible fluids deal with the polynomial dependence between $\mathbf{S}$ and $|\mathbf{D}\boldsymbol{u}|$. The analysis of nonhomogeneous Newtonian ($p = 2$ in (2)) fluids was investigated by Antontsev, Kazhikhov and Monakhov [**?**] in the seventies. P.L.

Lions in [35] presented the concept of renormalized solutions and obtained new convergence and continuity properties of the density.

The first result for unsteady flow of nonhomogenous non-Newtonian fluids goes back to Fernández–Cara [14], where existence of Dirichlet weak solutions was obtained for $p \geqslant 12/5$ if $d = 3$, later existence of space-periodic weak solutions for $p \geqslant 2$ with some regularity properties of weak solutions whenever $p \geqslant 20/9$ (if $d = 3$) was obtained by Guilién-González in [18]. Frehse and Růžička showed in [10] existence of a weak solution for generalized Newtonian fluid of power-law type for $p > 11/5$. Authors needed also existence of the potential of $\mathbf{S}$. Recent results concerning fluids where the growth condition is as in (2) for $p \geqslant 11/5$ belong to Frehse, Málek and Růžička [11]. The novelty of this paper is the consideration of the full thermodynamic model for a nonhomogeneous incompressible fluid. Particularly in [11, 10] the reader can find the concept of integration by parts formula, which we adapted to our case. Also more details concerning references can be found therein.

An example of a generalized Orlicz space is a generalized Lebesgue space, in this case $M(x, \boldsymbol{\xi}) = |\boldsymbol{\xi}|^{p(x)}$. These kind of spaces were applied in [45] to the description of flow of electrorheological fluid. The standard assumption in this work was $1 < p_0 \leqslant p(x) \leqslant p_\infty < \infty$, where $p \in C^1(\Omega)$ is a function of an electric field E, i.e. $p = p(|E|^2)$, and $p_0 > \frac{3d}{d+2}$ in case of steady flow, where $d \geqslant 2$ is the space dimension. The $\triangle_2$–condition is then satisfied and consequently the space is reflexive and separable. One of the main problems in our model is that the $\triangle_2$–condition is not satisfied and we lose the above properties. Later in [3] the above result was improved by Lipschitz truncations methods for $L^{p(x)}$ setting for $\mathbf{S}$, where $\frac{2d}{d+2} < p(\cdot) < \infty$ was log-Hölder continuous and $\mathbf{S}$ was strongly monotone.

First results concerning non-Newtonian fluid with the assumption that $\mathbf{S}$ is strictly monotone and satisfies conditions (3) and monotonicity assumption on $\mathbf{S}$ were established by Gwiazda et al. [19] for the case of unsteady flow. The stronger assumption on $\mathbf{S}$ was crucial for the applied tools (Young measures). This restriction was abandoned in [53] by Wróblewsk-Kamińska for the case of steady flow and in [21] by Gwiazda et al. for unsteady flow. The authors used generalization of Minty trick for non-reflexive spaces. The above existence results were established for $p \geqslant 11/5$ in [21], but without including in the system the dependence on density.

In order to present some of well known results concerning application of Orlicz space setting we recall some existing analytical results concerning the abstract parabolic problems in non-separable Orlicz spaces with zero Dirichlet boundary condition. Donaldson in [5] assumed that the nonlinear operator is an elliptic second-order, monotone operator in divergence form. The growth and coercivity conditions were more general than the standard growth conditions in L^p, namely the $\mathcal{N}$–function formulation was stated. Under the assumptions on the $\mathcal{N}-$function M: $\xi^2 << M(|\xi|)$ (i.e., ξ^2 grows essentially less rapidly than $M(|\xi|)$) and M^* satisfies the Δ_2-condition, existence result to parabolic equation was established. These restrictions on the growth of M were abandoned in [8].

The review paper [39] by Mustonen summarises the monotone-like mappings techniques in Orlicz and Orlicz–Sobolev spaces. The authors need essential modifications of such notions as: monotonicity, pseudomonotonicity, operators of type (M), (S_+), et al. The reason is that Orlicz–Sobolev spaces are not reflexive in general. Moreover, the nonlinear differential operators in divergence form with standard growth conditions are neither bounded nor everywhere defined.

A general class of elliptic equations with right hand side integrable only in L^1 space was considered in [25] by Gwiazda et al. (parabolic case in [26]). We extend there the theory of renormalized solutions to the setting of Orlicz spaces given by a nonhomogeneous anisotropic $\mathcal{N}$–function with non polynomial upper bound. See also [27].

4 Notation

Within the whole thesis we will use the following notation: Ω stands for bounded domain in $\mathbb{R}^d$, $(0, T)$ is a time interval and $Q := (0, T) \times \Omega$.

The following notation for function spaces is introduced $\mathcal{D}(\Omega)$ is a set of $C^\infty(\Omega)$-functions with compact support contained in Ω, $\mathcal{V}(\Omega)$ denotes functions $\varphi \in \mathcal{D}(\Omega)$ such that $\operatorname{div}\varphi = 0$. Moreover, by $L^p, W^{1,p}$ we mean the standard Lebesgue and Sobolev spaces respectively and $L^2_{\mathrm{div}}(\Omega)$

is the closure of $\mathcal{V}$ w.r.t. the $\|\cdot\|_{L^2}$-norm and $W^{1,p}_{0,\mathrm{div}}(\Omega)$ is the closure of $\mathcal{V}$ w.r.t. the $\|\nabla(\cdot)\|_{L^p}$-norm. Let $W^{-1,p'} = (W^{1,p}_0)^*$, $W^{-1,p'}_{\mathrm{div}} = (W^{1,p}_{0,\mathrm{div}})^*$. By p' we mean the conjugate exponent to p, namely $\frac{1}{p} + \frac{1}{p'} = 1$. We will use $C_{\mathrm{weak}}([0,T]; L^2(\Omega))$ in order to denote the space of functions $\boldsymbol{u} \in L^\infty(0,T; L^2(\Omega))$ which satisfy $(\boldsymbol{u}(t), \varphi) \in C([0,T])$ for all $\varphi \in L^2(\Omega)$. If X is a Banach space of scalar functions, then X^d or $X^{d\times d}$ denotes the space of vector- or tensor-valued functions where each component belongs to X. The symbols $L^p(0,T;X)$ and $C([0,T];X)$ mean the standard Bochner spaces. Finally, we recall that the Nikolskii space $N^{\alpha,p}(0,T;X)$ corresponding to the Banach space X and the exponents $\alpha \in (0,1)$ and $p \in [1,\infty]$ is given by $N^{\alpha,p}(0,T;X) := \{f \in L^p(0,T;X) : \sup_{0<h<T} h^{-\alpha}\|\tau_h f - f\|_{L^p(0,T-h;X)} < \infty\}$, where $\tau_h f(t) = f(t+h)$ for a.a. $t \in [0, T-h]$.

By (a,b) we mean $\int_\Omega a(x)\cdot b(x)dx$ and $\langle a,b\rangle$ denotes the duality pairing. By "$\cdot$" we denote the scalar product of two vectors and ":" stands for the scalar product of two tensors.

4.1 Orlicz spaces. Notion and propertioes

Definition 4.1 *Let Ω be a bounded domain in $\mathbb{R}^d$, a function $M : \Omega \times \mathbb{R}^n \to \mathbb{R}_+$ is said to be an* $\mathcal{N}-$function *if it satisfies the following conditions: M is a Carathéodory function, $M(x,\mathbf{K}) = 0$ if and only if $\mathbf{K} = 0$, $M(x,\mathbf{K}) = M(x,-\mathbf{K})$ a.e. in Ω, $M(x,\mathbf{K})$ is a convex function w.r.t. $\mathbf{K}$, and $\lim_{|\mathbf{K}|\to 0} \frac{M(x,\mathbf{K})}{|\mathbf{K}|} = 0$ for a.a. $x \in \Omega$, $\lim_{|\mathbf{K}|\to\infty} \frac{M(x,\mathbf{K})}{|\mathbf{K}|} = \infty$ for a.a. $x \in \Omega$.*

The *complementary function* M^* to a function M is defined by

$$M^*(x,\mathbf{L}) = \sup_{\mathbf{K}\in\mathbb{R}^n} (\mathbf{K} : \mathbf{L} - M(x,\mathbf{K})) \tag{6}$$

for $\mathbf{L} \in \mathbb{R}^n$, $x \in \Omega$. The complementary function M^* is also an $\mathcal{N}-$function.

The *generalized Orlicz class* $\mathcal{L}_M(Q;\mathbb{R}^n)$ is the set of all measurable functions $\mathbf{K} : Q \to \mathbb{R}^n$ such that

$$\int_Q M(x,\mathbf{K}(t,x))\,\mathrm{d}x\mathrm{d}t < \infty.$$

The *generalized Orlicz space* $L_M(Q;\mathbb{R}^n)$ is defined as the set of all measurable functions $\mathbf{K} : Q \to \mathbb{R}^n$ which satisfy

$$\int_Q M(x,\lambda\mathbf{K}(t,x))\,\mathrm{d}x\mathrm{d}t \to 0 \quad \text{as}\ \lambda \to 0.$$

The generalized Orlicz space is a Banach space with respect to the Luxemburg norm

$$\|\mathbf{K}\|_M = \inf\left\{\lambda > 0 \mid \int_Q M\left(x, \frac{\mathbf{K}(t,x)}{\lambda}\right)\mathrm{d}x\mathrm{d}t \leqslant 1\right\}.$$

Let us denote by $E_M(Q;\mathbb{R}^n)$ the closure of all measurable, bounded functions on Q in $L_M(Q;\mathbb{R}^n)$. The space $L_{M^*}(Q;\mathbb{R}^n)$ is the dual space of $E_M(Q;\mathbb{R}^n)$. It is easy to see that $E_M \subseteq \mathcal{L}_M \subseteq L_M$.

The functional $\varrho(\mathbf{K}) = \int_Q M(x,\mathbf{K}(x))\,\mathrm{d}x\mathrm{d}t$ is a modular in the space of measurable functions $\mathbf{K} : Q \to \mathbb{R}^n$. A sequence $\{\mathbf{z}^j\}_{j=1}^\infty$ converges modularly to z in $L_M(Q;\mathbb{R}^n)$ if there exists $\lambda > 0$ such that $\int_Q M\left(x, \frac{\mathbf{z}^j - \mathbf{z}}{\lambda}\right)\mathrm{d}x\mathrm{d}t \to 0$ as $j \to \infty$. We will write $\mathbf{z}^j \xrightarrow{M} \mathbf{z}$ for the modular convergence in $L_M(Q;\mathbb{R}^n)$.

We say that an $\mathcal{N}-$function M satisfies Δ_2*–condition* if for some nonnegative, integrable on Ω function g_M and a constant $C_M > 0$

$$M(x,2\mathbf{K}) \leqslant C_M M(x,\mathbf{K}) + g_M(x) \quad \text{for all}\ \mathbf{K} \in \mathbb{R}^n \ \text{and a.a.}\ x \in \Omega. \tag{7}$$

If this condition fails we lose numerous properties of the space $L_M(Q;\mathbb{R}^n)$ like separability, density of C^∞-functions, reflexivity (even in simpler case for $M(x,\mathbf{K}) = M(|\mathbf{K}|)$).

Depending on the considered problem we consider the $\mathcal{N}$–function of various form: in full generality like in definition above or $M(x,\mathbf{K}) = M(\mathbf{K})$ or $M(x,\mathbf{K}) = M(|\mathbf{K}|)$.

5 Generalized Navier-Stokes system

As a first we recall the result of existence of weak solutions to unsteady flow of non-Newtonian incompressible nonhomogenous (we do not assume that density is constant) fluids with nonstandard growth conditions of the stress tensor [54] by Wróblewska-Kamińska. We are motivated by the fluids of anisotropic behaviour and characterised by rapid shear thickness. These studies extend the existence theory for flows of non-Newtonian incompressible fluids to a more general class than polynomial growth conditions [11, 10] by formulating the problem in nonhomogeneous in space (x–dependent) anisotropic Orlicz setting as in [19, 21, 53]. Moreover, we want to complete the results the reader can find therein by including continuity equation $(1)_1$ to the considered system and dependence of $\mathbf{S}$ on density of the fluid, namely we do not assume that density is constant. Additionally we are able to obtain better regularity of solution in time than in [11, 10, 19, 21, 53], namely in the Nikolskii space.

In particular we assume also that the stress tensor $\mathbf{S} : (0,T)\times\Omega\times\mathbb{R}_+\times\mathbb{R}^{3\times3}_{\mathrm{sym}}\to\mathbb{R}^{3\times3}_{\mathrm{sym}}$ satisfies ($\mathbb{R}^{3\times3}_{\mathrm{sym}}$ stands for the space of 3×3 symmetric matrices):

S1 $\mathbf{S}(t,x,\varrho,\mathbf{K})$ is a Carathéodory function (i.e., measurable function of t, x for all $\varrho>0$ and $\mathbf{K}\in\mathbb{R}^{3\times3}_{\mathrm{sym}}$ and continuous function of ϱ and $\mathbf{K}$ for a.a. $x\in\Omega$) and $\mathbf{S}(t,x,\varrho,\mathbf{0})=\mathbf{0}$.

S2 There exist a positive constant c_c, $\mathcal{N}-$functions $M:\Omega\times\mathbb{R}^{3\times3}\to\mathbb{R}_+$ and M^* which denotes the complementary function to M such that for all $\mathbf{K}\in\mathbb{R}^{3\times3}_{\mathrm{sym}}$, $\varrho>0$ and a.a. $t,\ x\in Q$ it holds

$$\mathbf{S}(t,x,\varrho,\mathbf{K}):\mathbf{K}\geqslant c_c\{M(x,\mathbf{K})+M^*(x,\mathbf{S}(t,x,\varrho,\mathbf{K}))\}. \tag{8}$$

S3 $\mathbf{S}$ is monotone, i.e. for all $\mathbf{K}_1,\mathbf{K}_2\in\mathbb{R}^{3\times3}_{\mathrm{sym}}$, $\varrho>0$ and a.a. $x\in\Omega$

$$[\mathbf{S}(t,x,\varrho,\mathbf{K}_1)-\mathbf{S}(t,x,\varrho,\mathbf{K}_2)]:[\mathbf{K}_1-\mathbf{K}_2]\geqslant 0.$$

Definition 5.1 *We call the pair ϱ, $\boldsymbol{u}$ a* weak solution *to (1) if*

$$\begin{aligned}
&0<\varrho_*\leqslant\varrho(t,x)\leqslant\varrho^*\quad\textit{for a.a. }(t,x)\in Q,\\
&\varrho\in C([0,T];L^q(\Omega))\quad\textit{for arbitrary }q\in[1,\infty),\\
&\partial_t\varrho\in L^{5p/3}(0,T;(W^{1,5p/(5p-3)})^*)\\
&\boldsymbol{u}\in L^\infty(0,T;L^2_{\mathrm{div}}(\Omega;\mathbb{R}^3))\cap L^p(0,T;W^{1,p}_{0,\mathrm{div}}(\Omega;\mathbb{R}^3))\cap N^{1/2,2}(0,T;L^2_{\mathrm{div}}(\Omega;\mathbb{R}^3)\\
&\mathbf{D}\boldsymbol{u}\in L_M(Q;\mathbb{R}^{3\times3}_{\mathrm{sym}})\quad\textit{and}\quad(\varrho\boldsymbol{u},\boldsymbol{\psi})\in C([0,T])\textit{ for all }\boldsymbol{\psi}\in L^2_{\mathrm{div}}(\Omega;\mathbb{R}^3)
\end{aligned}$$

$$\int_0^T\langle\partial_t\varrho,z\rangle-(\varrho\boldsymbol{u},\nabla_x z)\,\mathrm{d}t=0 \tag{9}$$

for all $z\in L^r(0,T;W^{1,r}(\Omega))$ with $r=5p/(5p-3)$, i.e.

$$\int_{s_1}^{s_2}\int_\Omega\varrho\partial_t z+(\varrho\boldsymbol{u})\cdot\nabla_x z\,\mathrm{d}x\mathrm{d}t=\int_\Omega\varrho z(s_2)-\varrho z(s_1)\,\mathrm{d}x$$

for all z smooth and s_1, $s_2\in[0,T]$, $s_1<s_2$ and

$$\begin{aligned}
&-\int_0^T\int_\Omega\varrho\boldsymbol{u}\cdot\partial_t\boldsymbol{\varphi}-\varrho\boldsymbol{u}\otimes\boldsymbol{u}:\nabla_x\boldsymbol{\varphi}+\mathbf{S}(t,x,\varrho,\mathbf{D}\boldsymbol{u}):\mathbf{D}\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t\\
&=\int_0^T\int_\Omega\varrho\boldsymbol{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t+\int_\Omega\varrho_0\boldsymbol{u}_0\cdot\boldsymbol{\varphi}(0)\,\mathrm{d}x\qquad\textit{for all }\boldsymbol{\varphi}\in\mathcal{D}((-\infty,T);\mathcal{V}),
\end{aligned}\tag{10}$$

and initial conditions are achieved in the following way

$$\lim_{t\to0^+}\|\varrho(t)-\varrho_0\|_{L^q(\Omega)}+\|\boldsymbol{u}(t)-\boldsymbol{u}_0\|^2_{L^2(\Omega)}=0\qquad\textit{for arbitrary }q\in[1,\infty). \tag{11}$$

Theorem 5.1 *Let M be an $\mathcal{N}-$function satisfying for some $\underline{c} > 0$, $\widetilde{C} \geqslant 0$ and $p \geqslant \frac{11}{5}$ the condition*

$$M(x, \boldsymbol{\xi}) \geqslant \underline{c}|\boldsymbol{\xi}|^p - \widetilde{C} \tag{12}$$

for a.a. $x \in \Omega$ and all $\boldsymbol{\xi} \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}$. Let us assume that the conjugate function

$$M^* \text{ satisfies the } \Delta_2 - \text{condition} \quad \text{and} \quad \lim_{|\boldsymbol{\xi}|\to\infty} \inf_{x\in\Omega} \frac{M^*(x,\boldsymbol{\xi})}{|\boldsymbol{\xi}|} = \infty. \tag{13}$$

Moreover, let $\mathbf{S}$ satisfy conditions **S1.-S3.** *and $\boldsymbol{u}_0 \in L^2_{\mathrm{div}}(\Omega;\mathbb{R}^3)$, $\varrho_0 \in L^\infty(\Omega)$ with $0 < \varrho_* \leqslant \varrho_0(x) \leqslant \varrho^* < +\infty$ for a.a. $x \in \Omega$ and $\boldsymbol{f} \in L^{p'}(0,T;L^{p'}(\Omega;\mathbb{R}^3))$. Then there exists a weak solution to (1).*

The all details on the proof of the above theorem can be found in [54], for some partial results and necessary methods see also [53, 21]. The first step of the proof of existence of a weak solution is the Galerikn approximation for the considered problem and existence of an approximate solution. The main difficulty then is to show the proper convergences in nonlinear terms. The result is achieved by a monotonicity method adapted to non-reflexive spaces [53, 21] and the compensated compactness method.

Using the result mentioned above in [56, 55] we consider the problem of motion of one or several nonhomogeneous rigid bodies immersed in a homogeneous non-Newtonian fluid occupying a bounded domain. Therefore the fluid flow in the system is of (1)-type which is completed with the equations describing the motion of rigid bodies. We use here the fact, proved by Starovoitov, that two rigid objects do not collide if they are immersed in a fluid of viscosity significantly increasing with increasing shear rate. The method we use in order to solve the problem is, in the first step, to replace the rigid object by a fluid of high viscosity becoming singular in the limit. This idea was developed by Hoffman [28] and San Marin at al. [46]. Since we consider an incompressible fluid, the existence and estimates for the pressure function are not crucial from the point of existence of weak solutions. This is due to the fact that in a weak formulation the pressure function disappears. In this case we have to localise the problem only in the fluid part of the system. Therefore we need to deliver the decomposition and local estimates also for the pressure function. To this end we use the Riesz transform which in general is not continuous from Orlicz space to itself (it is the case if the $\mathcal{N}$–function and its complementary satisfy the Δ_2–condition). Therefore the space where the part of our pressure function is regular is larger than the space containing the nonlinear viscous term. Moreover we are not able to use theorems of Marcinkiewicz type and interpolation theory in the same form as in Lebesgue or Sobolev spaces. For this reason the passage in terms associated with the regular part of the pressure function is much more demanding than in [13].

6 Generalized Stokes system

In the above two problems the presence of a convective term $\mathrm{div}(\boldsymbol{u}\otimes\boldsymbol{u})$ enforces at least polynomial growth of tensor $\mathbf{S}$ with respect to $\mathbf{D}\boldsymbol{u}$. With these assumptions we are able to investigate only the case of shear thickening fluids. This motivates us to consider the generalized Stokes system:

$$\begin{aligned} \partial_t \boldsymbol{u} - \mathrm{div}\mathbf{S}(t,x,\mathbf{D}\boldsymbol{u}) + \nabla p = \boldsymbol{f} &\quad \text{in } (0,T)\times\Omega, \\ \mathrm{div}\boldsymbol{u} = 0 &\quad \text{in } (0,T)\times\Omega, \\ \boldsymbol{u}(0,x) = \boldsymbol{u}_0 &\quad \text{in } \Omega, \\ \boldsymbol{u}(t,x) = 0 &\quad \text{on } (0,T)\times\partial\Omega, \end{aligned} \tag{14}$$

where $\Omega \subset \mathbb{R}^d$ is an open, bounded set with a sufficiently smooth boundary $\partial\Omega$, $(0,T)$ is the time interval with $T < \infty$, $Q = (0,T)\times\Omega$, $\boldsymbol{u} : Q \to \mathbb{R}^d$ is the velocity of a fluid and $p : Q \to \mathbb{R}$ the pressure, $\mathbf{S}+\mathbf{I}p$ is the Cauchy stress tensor. Here we assume that $\mathbf{S}$ satisfies the following conditions

(S1) $\mathbf{S}$ is a Carathéodory function (i.e., measurable w.r.t. t and x and continuous w.r.t. the last variable).

(S2) There exists a function $M : \mathbb{R}^{d\times d}_{\rm sym} \to \mathbb{R}_+$ and a constant $c > 0$ such that for all $\boldsymbol{\xi} \in \mathbb{R}^{d\times d}_{\rm sym}$

$$\mathsf{S}(t,x,\boldsymbol{\xi}) : \boldsymbol{\xi} \geqslant c(M(\boldsymbol{\xi}) + M^*(\mathsf{S}(t,x,\boldsymbol{\xi}))). \tag{15}$$

(S3) For all $\boldsymbol{\xi},\boldsymbol{\eta} \in \mathbb{R}^{d\times d}_{\rm sym}$ and for a.a. $t,\ x \in Q$

$$(\mathsf{S}(t,x,\boldsymbol{\xi}) - \mathsf{S}(t,x,\boldsymbol{\eta})) : (\boldsymbol{\xi} - \boldsymbol{\eta}) \geqslant 0.$$

The following result of existence of weak solutions to the generalized Stokes system with the nonlinear term having growth conditions prescribed by an anisotropic $\mathcal{N}-$function. Our main interest is directed to relaxing the assumptions on the $\mathcal{N}-$function and in particular to capture the shear thinning fluids with rheology close to linear. Additionally, for the purpose of the existence proof, a version of the Sobolev–Korn inequality in Orlicz spaces is proved in [22].

Theorem 6.1 ([22]) *Let condition D1. or D2. be satisfied*

(D1) Ω is a bounded star-shaped domain,

(D2) Ω is a bounded non-star-shaped domain and $\overline{m}(r) \leqslant c_m((\underline{m}(r))^{\frac{d}{d-1}} + |r|^2 + 1)$ for all $r \in \mathbb{R}_+$, and $\underline{m}$ satisfies Δ_2-condition.

Let M be an $\mathcal{N}-$function and S satisfy conditions (S1)-(S3). Then, for given $\boldsymbol{u}_0 \in L^2_{\rm div}(\Omega;\mathbb{R}^d)$ and $\boldsymbol{f} \in E_{\underline{m}^*}(Q;\mathbb{R}^d)$ there exists $\boldsymbol{u} \in Z^M_0$ such that

$$\int_Q -\boldsymbol{u}\cdot\partial_t\boldsymbol{\varphi} + \mathsf{S}(t,x,\mathsf{D}\boldsymbol{u})\cdot\mathsf{D}\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t = \int_Q \boldsymbol{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t - \int_\Omega \boldsymbol{u}_0\boldsymbol{\varphi}(0)\,\mathrm{d}x \tag{16}$$

for all $\boldsymbol{\varphi} \in \mathcal{D}(-\infty,T;\mathcal{V})$, where

$$\begin{aligned} Z^M_0 = &\{\boldsymbol{u} \in L^\infty(0,T;L^2_{\rm div}(\Omega;\mathbb{R}^d)),\, \mathsf{D}\boldsymbol{u} \in L_M(Q;\mathbb{R}^{n\times n}_{\rm sym}) \mid \exists\ \{\boldsymbol{u}^j\}^\infty_{j=1} \subset \mathcal{D}((-\infty,T);\mathcal{V}) : \\ &\boldsymbol{u}^j \xrightharpoonup{*} \boldsymbol{u} \text{ in } L^\infty(0,T;L^2_{\rm div}(\Omega;\mathbb{R}^d)) \text{ and } \mathsf{D}\boldsymbol{u}^j \xrightharpoonup{*} \mathsf{D}\boldsymbol{u} \text{ weakly star in } L_M(Q;\mathbb{R}^{d\times d}_{\rm sym})\}. \end{aligned} \tag{17}$$

and two functions $\underline{m}$, $\overline{m} : \mathbb{R}_+ \to \mathbb{R}_+$ are defined as as follows

$$\underline{m}(r) := \min_{\boldsymbol{\xi}\in\mathbb{R}^{d\times d}_{\rm sym},|\boldsymbol{\xi}|=r} M(\boldsymbol{\xi}), \qquad \overline{m}(r) := \max_{\boldsymbol{\xi}\in\mathbb{R}^{d\times d}_{\rm sym},|\boldsymbol{\xi}|=r} M(\boldsymbol{\xi}). \tag{18}$$

In particular the considerations of the above problem, which the reader can find in [22] by Gwiazda et al., allow us to investigate the case of shear thinning fluids, whose viscosity decreases when the shear rate increases. Let us notice that if we assume that the flow is slow, the density is constant and so the system stated in (1) can be reduced to (14). The problem is considered in anisotropic Orlicz spaces. In the proof we need to provide the type of the Korn-Sobolev inequality for anisotropic Orlicz spaces when the Δ_2–condition is not satisfied. We show also that the closure of smooth functions with compact support with respect to two topologies is equal: the convergence of symmetric gradients in modular and in weak star topology in Orlicz space. Then we are able to give the formula for integration by parts.

These studies consists of a new analytical approach to the existence problem. In the previous studies the main reason to assume that M^* satisfies the Δ_2-condition was providing that the solution is bounded in an appropriate Sobolev space $W^{1,q}(\Omega)$ which is compactly embedded in $L^2(\Omega)$. However, as a byproduct, we gained that $L_{M^*}(Q;\mathbb{R}^{d\times d}_{\rm sym}) = E_{M^*}(Q;\mathbb{R}^{d\times d}_{\rm sym})$ is a separable space. The naturally arising question is whether the existence of solutions can still be proved after omitting the convective term and relaxing the assumptions on M and M^*. The preliminary studies in this direction were done for an abstract parabolic equation, cf. [20]. Also the convergence of a full discretization of quasilinear parabolic equation can be found in [7] by Emmrich and Wróblewska-Kamińska. Theorem 6.1 give a non-trivial extension of these considerations for the system of equations.

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