TIME DELAY AND LAGRANGIAN DIFFERENCES FOR VISCOUS INCOMPRESSIBLE FLUID FLOW

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Abstract

The motion of a viscous incompressible fluid flow in bounded domains with a smooth boundary can be described by the nonlinear nonsteady Navier-Stokes equations. This description corresponds to the so-called Eulerian representation approach. We develop an approximation method for the Navier-Stokes system by a suitable coupling of the Eulerian and the Lagrangian representation of the flow, where the latter is defined by the trajectories of the particles of the fluid. The method leads to a sequence of uniquely determined approximate solutions with a high degree of regularity, which contains a convergent subsequence with limit function v such that v is a weak solution of the Navier-Stokes system in the sense of Leray-Hopf.

Keywords: Navier-Stokes equations, time delay, Lagrangian differences.

1 Introduction

We consider the nonstationary nonlinear Navier-Stokes equations

$$\partial_t v - \Delta v + v \cdot \nabla v + \nabla p = 0, \quad \nabla \cdot v = 0, \qquad v = 0 \text{ on } \partial\Omega, \quad v = v_0 \text{ for } t = 0 \tag{1}$$

in a bounded cylindrical domain $(0, T) \times \Omega$, where $T > 0$ and $\Omega \subset \mathbb{R}^3$ is a bounded domain with a smooth boundary $\partial\Omega$. These equations describe the motion of a viscous incompressible fluid contained in Ω for $0 < t < T$: $v = (v_1, v_2, v_3)$ represents the velocity of a fluid particle and p the pressure at time t at position $x \in \Omega$. In the present case the external force is assumed to be conservative, and the kinematic viscosity is normed to one, thus only the initial velocity v_0 is given.

Besides the description of a flow by its velocity v there is another approach, using the Lagrangian coordinates $X(t, 0, x_0) \in \Omega$ [1]. Here the function $t \to x(t) = X(t, 0, x_0)$ denotes the trajectorie of a fluid particle, which at initial time $t = 0$ is located at $x_0 \in \Omega$. This approach has been used for the treatment of Navier-Stokes and transport equations ([3]) and is of great importance for the numerical computation of a flow involving different media with interfaces. Both representations are correlated by the equations

$$\dot{x}(t) = v(t, x(t)), \quad x(0) = x_0\,, \tag{2}$$

which is an initial value problem for ordinary differential equations, if the velocity v is known.

Temam [12] has shown, that the system (2) can be solved, even if v is a solution of (1) in a weak sense only (Hopf [8]). Moreover, in the paper jointly with Foias and Guillopé [4] also the volume conserving property of the mappings $X(t, 0, \cdot) : \Omega \to \Omega$ has been proved for this case. However, for an efficient numerical treatment of (2) it would be highly desirable to work with Navier-Stokes solutions v, which are as smooth as possible, at least at initial time $t = 0$. By results of Heywood, Rannacher [7] and Temam [11] this leads to the compatibility conditions: For the strong $H^3(\Omega)$-continuity of any Navier-Stokes solution at $t = 0$, which at least would imply the unique solvability of (2) and some regularity of the solution, the corresponding initial velocity v_0 has to satisfy a nonlocal compatibility condition, which in general is uncheckable.

In the present paper we construct an energy conserving Lagrangian difference quotient, which approximates the nonlinear convective term $v \cdot \nabla v$ in (1). By a suitable time delay it is possible to determine the trajectories of the fluid particles from the velocity field and vice versa successively, such that the resulting equations can be solved for all times. A special initial construction of compatible data ensures that the corresponding solution is uniquely determined and strongly $H^4(\Omega)$-continuous uniformly in time. Passing to the limit for the Lagrangian difference quotient the following convergence result can be proved: There always exists a subsequence of the solutions, which for all time converges against a weak solution of the Navier-Stokes equations (1) in the sense of Leray-Hopf [8].

Let us outline our notation: $I \subset \mathbb{R}$ always denotes a compact interval and $\Omega \subset \mathbb{R}^3$ a bounded domain with boundary $\partial\Omega$ of class C^4 and closure $\overline{\Omega} := \Omega \cup \partial\Omega$. In the following we use the same symbols for scalar and vector valued functions (always real) as well as for the corresponding function spaces and norms. We need the spaces $L_p(\Omega)(1 \le p \le \infty)$, $C^m(\Omega)$, $C^m(\overline{\Omega})$, $C_0^m(\Omega)$, and the Sobolev (Hilbert) spaces $H^m(\Omega)$, $H_0^m(\Omega)$ $(m = 0, 1, \ldots; H_0^0(\Omega) = H^0(\Omega) = L_2(\Omega))$. By $H(\Omega)$ and $V(\Omega)$ we denote the closure of

$$D(\Omega) := \{u \in C_0^\infty(\Omega) | \operatorname{div} u = 0\}$$

in $L_2(\Omega)$ and $H^1(\Omega)$, respectively. Moreover we use the B-valued spaces $C^m(I, B)$ $(m = 0, 1, \ldots)$ and $L_p(a, b, B)$, where $a, b \in \mathbb{R}$ $(a < b)$ and B is any of the spaces above. Instead of $C^0(\ )$ we write $C(\ )$, and sometimes we supress the domain Ω in the function spaces: $V := V(\Omega)$, $C^1(I, L_p) := C^1(I, L_p(\Omega))$, ... The norm in $L_p(\Omega)$ and in $H^m(\Omega)$ is denoted by $||\cdot||_{0,p}$ and $||\cdot||_m$, respectively, where in particular we set $||\cdot|| := ||\cdot||_{0,2} = ||\cdot||_0$ and

$$||\cdot||_\infty := ||\cdot||_{0,\infty} := \operatorname*{ess\,sup}_{x\in\Omega} |\cdot(x)|$$

with the Euclidian norm $|\cdot|$. For $v = (v_1, v_2, v_3)$ and $u = (u_1, u_2, u_3)$ we use

$$(v, u) := \int_\Omega \sum_{i=1}^3 v_i(x) u_i(x) dx$$

as scalar product in $L_2(\Omega)$. The mapping $P : L_2(\Omega) \to H(\Omega)$ denotes the Helmholtz projection such that

$$L_2(\Omega) = H(\Omega) \oplus \{u \in L_2(\Omega) | u = \operatorname{grad} p, p \in H^1(\Omega)\}.$$

With $D_i (i = 1, 2, 3)$ as the partial derivative with respect to x_i we set $\nabla := (D_1, D_2, D_3) = \operatorname{grad}$ and define

$$\nabla \cdot v := \sum_{i=1}^3 D_i v_i = \operatorname{div} v,\ v \cdot \nabla u := \left(\sum_{i=1}^3 v_i D_i u_j\right)_j,\ \nabla v := (D_j v_k)_{kj}$$

and $\nabla^2 v := (D_1 D_j v_k)_{kj1}$. In $V(\Omega)$ and $H_0^1(\Omega)$ we mostly use

$$(\nabla v, \nabla u) := \sum_{j=1}^3 (D_j v, D_j u)$$

and $||\nabla v|| := (\nabla v, \nabla v)^{\frac{1}{2}}$ as scalar product and norm, respectively.

2 Lagrangian Differences

Let $v \in C(I, H^m(\Omega) \cap V(\Omega)), m \in \{3, 4\}$ and consider for $(s, x_s) \in I \times \overline{\Omega}$ the equations

$$\dot{x}(t) = v(t, x(t)),\ x(s) = x_s\,. \tag{3}$$

Because v vanishes on $I \times \partial\Omega$ and, as $H^3(\Omega)$-continuous function, certainly satisfies a uniqueness condition for (3), it follows that the solution $t \to x(t) =: X(t, s, x_s)$ exists in the whole interval I and is uniquely determined there. Due to the uniqueness, the mappings

$$X(t, s) := X(t, s, \cdot) : \begin{cases} \overline{\Omega} \to \overline{\Omega} \\ x \to X(t, s, x) \end{cases} \tag{4}$$

satisfy

$$X(t,s) \circ X(s,r) := X(t,s,X(s,r,\cdot)) = X(t,r)$$

for all $t,s,r \in I$, and, in particular, $X(t,s)$ is a C^{m-2}-diffeomorphism on $\overline{\Omega}$ with inverse mapping

$$(X(t,s))^{-1} = X(s,t).$$

Since $v = 0$ on $I \times \partial\Omega$ implies $X(t,s,\Omega) = \Omega$, and since div $v = 0$ in $I \times \Omega$, we obtain from Liouville's differential equation

$$\partial_t \det\nabla X(t,s,x) = \mathrm{div}_X v(t,X(t,s,x)) \cdot \det\nabla X(t,s,x)$$

for the Jacobian the identity

$$\det\nabla X(t,s,x) = \det\nabla X(s,s,x) = \det\nabla x = 1.$$

This volume conserving property leads to

$$||v(t,X(s,r,\cdot))||_{0,p} = ||v(t,\cdot)||_{0,p} \quad (1 \leq p \leq \infty)\,, \tag{5}$$

which holds for all $t,s,r \in I$.

In order to approximate the nonlinear convective term $v \cdot \nabla v$ in (1) we return to its physical origin. The term arises from the total (material) derivative of the velocity v, and thus we use total differences for approximation:

2.1 Definition. *Let $t,s,s+h \in I$ $(h > 0), x \in \overline{\Omega}$, and assume $v \in C(I, H^3(\Omega) \cap V(\Omega))$. Then we call*

$$\frac{1}{h}\{v(t,X(s+h,s,x)) - v(t,x)\}, \quad \frac{1}{h}\{v(t,x) - v(t,X(s,s+h,x))\}\,, \tag{6}$$

$$\frac{1}{2h}\{v(t,X(s+h,s,x)) - v(t,X(s,s+h,x))\} \tag{7}$$

an upwards taken, a backwards taken, and a central Lagrangian difference quotient, respectively.

For $h \to 0$, every quotient above converges to $v(s,x) \cdot \nabla v(t,x)$. For instance, using (3) we obtain

$$\begin{aligned} v(t,X(s+h,s,x)) - v(t,x) &= v(t,X(s+h,s,x)) - v(t,X(s,s,x)) \\ &= \int_s^{s+h} \partial_r X(r,s,x) \cdot \nabla v(t,X(r,s,x))dr \\ &= \int_s^{s+h} (v(r) \cdot \nabla v(t)) \circ X(r,s,x)dr\,, \end{aligned}$$

and a mean value theorem yields the assertion. But in contrast to (6), only for the central quotient (7) an $L_2(\Omega)$-orthogonality relation holds. Using the volume conserving property of the mappings X in the form

$$\begin{aligned} (v \circ X - v \circ X^{-1}, v) &= (v \circ X, v) - (v \circ X^{-1}, v) \\ &= (v \circ X \circ X^{-1}, v \circ X^{-1}) - (v \circ X^{-1}, v) \\ &= (v, v \circ X^{-1}) - (v \circ X^{-1}, v) = 0\,, \end{aligned}$$

we find:

2.2 Lemma. *Under the assumptions of Definition 2.1 we have*

$$\left(\frac{1}{2h}\{v(t,X(s+h,s,\cdot)) - v(t,X(s,s+h,\cdot))\}, \quad v(t,\cdot)\right) = 0\,. \tag{8}$$

The $L_2(\Omega)$-orthogonality relation (8) is an analogon to the relation

$$(u \cdot \nabla w, w) = 0 \; (u \in V(\Omega), w \in H_0^1(\Omega)),$$

which is used by Hopf [8] to show the global existence of weak Navier-Stokes solutions. Thus it follows from the consideration above, that the central quotient (7) only leads to an energy conserving approximation procedure.

In order to avoid fixpoint considerations (both the velocity and the corresponding trajectories are not known), we additionally use a time delay and substitute the convective term $v(t,x)\cdot\nabla v(t,x)$ by centered differences of the form

$$\frac{1}{2h}\{v(t, X(s+h, s, x)) - v(t, X(s, s+h, x))\}$$

with $s + h < t$. This leads to an approximation, where the velocity and the trajectories have to be determined from each other successively. Concrete we choose the following scheme:

Assume $T > 0$ and $N \in \mathbb{N}$ $(N \geq 2)$. Define $h := \frac{T}{N} > 0$ and let $t_i := ih$ $(i = -2, -1, \ldots, N)$ be a grid on $[-2h, T]$. Now for

$$(t, x) \in [t_k, t_{k+1}] \times \overline{\Omega} \quad (k = 0, 1, \ldots, N-1) \;\text{ replace }\; v(t, x) \cdot \nabla v(t, x)$$

by

$$\begin{aligned} Z_h v(t,x) \;&:=\; Z_h^k v(t,x) \qquad (9)\\ &:=\; \frac{t - t_k}{2h^2}\{v(t, X(t_k, t_{k-1}, x)) - v(t, X(t_{k-1}, t_k, x))\} + \\ &\quad\;\; \frac{t_{k+1} - t}{2h^2}\{v(t, X(t_{k-1}, t_{k-2}, x)) - v(t, X(t_{k-2}, t_{k-1}, x))\}\,. \end{aligned}$$

2.3 Remark. (a) The determination of $v(t)$ for $t \in [t_0, t_1]$ requires an initial construction, which is carried out in the next section.
(b) In (9) the mappings $X : \overline{\Omega} \to \overline{\Omega}$ do not depend on $t \in [t_k, t_{k+1}]$, which means a simplification from the numerical point of view. Nevertheless, the continuity on $[0, T]$ of the functions $Z_h v(\cdot, x)$ does still hold and ensures the global existence of a unique solution in the next section.

3 Global Existence, Uniqueness, Compatibility

It is known ([7], [9], [11]), that the compatibility condition, which has to be satisfied by any solution of (1) in case of strong $H^3(\Omega)$-continuity at $t = 0$, cannot be proved in general, if the corresponding initial velocity v_0 is given. But still, following a hint of Solonnikov, we can construct an initial velocity v_0 in such a way, that this condition is fulfilled, and in the present case of scheme (9), moreover, this construction is unique. To do so, let us replace in (1) the convective term by (9) and the initial condition $v(0) = v_0$ by $\partial_t v(0) = a_0$, obtaining at $t = 0$ in Ω the stationary (projected) equations

$$P(a_0 - \Delta v_0 + \frac{1}{2h}\{v_0 \circ X(-h, -2h) - v_0 \circ X(-2h, -h)\}) = 0 \qquad (10)$$

with some prescribed initial acceleration a_0. The construction is now stated in

3.1 Lemma. *Assume* $T > 0, m \in \{3, 4\}, u \in C([-T, 0], H^m(\Omega) \cap V(\Omega))$, *and* $a_0 \in H^{m-2}(\Omega) \cap V(\Omega)$. *Let* $N \in \mathbb{N}$ $(N \geq 2)$ *and define* $h := \frac{T}{N} > 0$. *Then*:

(a) *Replacing* v *by* u *in* (3), *the mappings* $X(-h,-2h)$ *and* $X(-2h,-h)$ *in* (10) *are uniquely defined by* (4).

(b) *There exists a uniquely determined solution* $v_0 \in H^m(\Omega) \cap V(\Omega)$ *of* (10).

(c) *The function* v *given by*

$$v(t) := \begin{cases} u(t) & \text{for } t \in [-T,-h] \\ \frac{1}{h}\{(t+h)v_0 - tu(-h)\} & \text{for } t \in [-h,0] \end{cases} \tag{11}$$

belongs to $C([-T,0], H^m(\Omega) \cap V(\Omega))$, *and hence the mappings* $X(0,-h)$ *and* $X(-h,0)$ *in* (9) *are uniquely defined by* (3), (4).

3.2 Theorem. *Assume that the initial construction is carried out as in* Lemma 3.1, *and that, in particular,* $v_0 \in H^m(\Omega) \cap V(\Omega)$ *denotes the unique solution of* (10). *Then there exist unique functions* $v \in C^j([0,T],\ H^{m-2j}(\Omega) \cap V(\Omega))$, $j \in \{0,1\}$, *and* $\nabla p \in C([0,T], H^{m-2}(\Omega))$ *satisfying in* $(0,T) \times \Omega$

$$\partial_t v - \Delta v + Z_h v + \nabla p = 0, \qquad \nabla \cdot v = 0, \qquad v = 0 \text{ on } \partial\Omega, \qquad v = v_0 \text{ for } t = 0\,, \tag{12}$$

where $Z_h v$ *is defined by* (9). *For* $t \in [0,T]$ *the solution* v *statisfies the energy equation*

$$||v(t)||^2 + 2\int_0^t ||\nabla v(s)||^2 ds = ||v_0||^2\,. \tag{13}$$

3.3 Remark. The global construction in the proof above works without any smallness assumptions for the prescribed initial acceleration a_0 and the function u in Lemma 3.1. Due to appearing nonlinearities, a similar construction to fulfill higher order compatibility conditions (cf. Temam [11]) without any smallness assumptions does not seem to be possible up to now.

4 A Leray-Hopf Construction

In the general threedimensional situation, the only solutions to the Navier-Stokes equations (1), which exist for all times, are solutions in a weak sense (Hopf [8]; compare also Temam [10]). Let us recall:

4.1 Definition. *Assume* $T > 0$ *and* $v_0 \in H(\Omega)$. *Then a function* $v \in L_2(0,T,V(\Omega)) \cap L_\infty(0,T,H(\Omega))$ *is called a weak solution of the Navier-Stokes equations* (1) *with initial value* v_0, *if* $v : [0,T] \to H(\Omega)$ *is weakly continuous, if* $||v(t) - v_0|| \to 0$ *for* $t \to 0$, *and if for all* $\Phi \in C_0^\infty((0,T) \times \Omega)$ *with* $\Phi(t) \in D(\Omega)$ $(0 < t < T)$ *the identity*

$$\int_0^T \{-(v(t), \partial_t \Phi(t)) + (\nabla v(t), \nabla \Phi(t)) - (v(t) \cdot \nabla \Phi(t), v(t))\} dt = 0 \tag{14}$$

is satisfied.

We can show, that those solutions can be constructed by the solution of the system (12), if in Theorem 3.2 for $N \to \infty$ (T remains fixed) the stepsize $h := \frac{T}{N} > 0$ goes to zero. To express the dependence of N, in the following we write h_N, v^N, v_0^N instead of h, v, v_0. Our main result is now stated in

4.2 Theorem. *Let* $T > 0$ *be fixed, and let* $h_N := \frac{T}{N} > 0$ *for* $N = 2, 3, \ldots$ *As constructed in* Lemma 3.1 *and* Theorem 3.2, *respectively, let* v_0^N *and* v^N *denote the initial value and the solution of the corresponding equations* (12). *Then there exists a convergent subsequence* $(v_0^{N_k})_k$ *of* $(v_0^N)_N$

with limit v_0 and a convergent subsequence $(v^{N_k})_k$ of $(v^N)_N$ with limit v such that v is a weak solution of the Navier-Stokes equations (1) with initial value v_0 and satisfies for $t \in [0,T]$ the energy inequality

$$||v(t)||^2 + 2\int_0^t ||\nabla v(s)||^2 ds \leq ||v_0||^2 \,. \tag{15}$$

Choosing $u = 0$ in Lemma 3.1, the system (10) reduces to the Stokes equations. Because its unique solution v_0 does not depend on N, in Theorem 4.2 we have $v_0^N = v_0$ for all $N \in \mathbb{N}$ ($N \geq 2$). Let us conclude with a final consideration concerning strong solutions of (1):

4.3 Remark. In Theorem 4.2 for all $N \in \mathbb{N}$ ($N \geq 2$) the same function u is used for the initial construction in Lemma 3.1. The statement of Theorem 4.2 remains valid, if the function u depends on N as follows: Let $u := u^N$ be given for some $N \geq 2$. Then define v^N by (11) and choose $u^{N+1} := v^N$ in the next step. Now, under this modification, let v_0 be any accumulation point of the sequence $(v_0^N)_N$ mentioned in Theorem 4.2, and let v be the corresponding unique strong solution of (1), existing on a (possibly small) time interval $[0, T^*]$, $0 < T^* \leq T$ ([5], [6]). Then it can be shown by the same methods as in the proof above that v belongs to $C^j([0,T^*],\ H^{m-2j}(\Omega) \cap V(\Omega))$, $j \in \{0,1\}$, with $\partial_t^2 v \in C([0,T^*], H(\Omega))$, and that v satisfies $\partial_t v(0) = a_0$.

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