(AN)ISOTROPY ANALYSIS OF TURBULENCE

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Abstract

Basic ideas of (an)isotropy analysis and rating of turbulent flows are presented oriented on point velocity data analysis. Two types of anisotropy invariant maps are proposed. Practical example is shown.

Keywords: turbulence, statistics, isotropy

1 Introduction

Our knowledge of turbulent flows is still far from being satisfactory. The most theories apply only to “well developed” turbulent flow which is characterized by unrealistic features. For example the Kolmogorov theories hold for isotropic, homogeneous turbulent field characterized by turbulence Reynolds number approaching infinity. Homogeneity and isotropy are considered in statistical sense.

Any real case is characterized by some departures from that ideal case. It is essential to quantify the above mentioned features, to verify relevance of the theories for the given case.

In the presented paper we will concentrate on quantification of isotropy of the turbulence, characterized by time series of all 3 velocity component in the given point in space. The isotropy is evaluated in statistical sense on the basis of variances and covariances analysis. Using this data we could evaluate the complete tensor of Reynolds stress. The subsequent (an)isotropy analysis is oriented on this kind of input information.

2 Basic definitions

Dynamics and 3D structure are the basic attributes of a turbulent flow, however there are some others as range of scales, vorticity, dissipativity etc. Statistical approach to turbulence relies on the velocity field information in a single point in space. Then the relevant information is contained in Reynolds stress tensor.

2.1 Reynolds stress tensor

The Reynolds stress is the component of the total stress tensor in a fluid obtained from the averaging operation over the Navier-Stokes equations to account for turbulent fluctuations in fluid momentum. The Reynolds stress tensor is defined as

$$\tau_{ij} = -\rho\,\overline{u_i u_j}\,, \tag{1}$$

where ρ stands for fluid density and u_i is i-th velocity component fluctuation. From the mathematical point of view the Reynolds stress tensor is nothing but a symmetrical tensor of second order – matrix. From linear algebra it is known that any symmetrical matric could be decomposed into isotropic τ_{ij}^I and anisotropic τ_{ij}^A parts:

$$\tau_{ij} = \tau_{ij}^I + \tau_{ij}^A\,. \tag{2}$$

Isotropy is here considered as independence on the direction in physical space described by Cartesian coordinate system. The decomposition could be performed in the following way:

$$\tau_{ij}^{I}=\frac{1}{3}\tau_{kk}\ \delta_{ij};\quad \tau_{ij}^{A}=\tau_{ij}-\tau_{ij}^{I}. \tag{3}$$

Note that isotropic part is equal to unity matrix multiplied by a constant equal to one third of turbulent kinetic energy or a matrix trace if you like.

For further analysis the anisotropic part is considered in its nondimensional form

$$b_{ij}=\frac{\tau_{ij}^{A}}{\tau_{kk}}=\frac{\tau_{ij}}{\tau_{kk}}-\frac{\delta_{ij}}{3}. \tag{4}$$

The nondimensional anisotropic part of Reynolds stress tensor, or simply *anisotropy tensor* b_{ij}, is considered to be the fundamental characteristic of the turbulence anisotropy in the given point. Of course, if this tensor is vanishing, the flow is considered to be perfectly isotropic.

The anisotropy tensor is nondimensional, its value is not affected by any multiplicative constant applied to the stress tensor. This implies that this is identical for Reynolds stress tensor and correlation matrix.

In following part of the paper we will go through the detailed analysis and quantification of this tensor significance.

2.2 Anisotropy invariants

The nondimensional anisotropic part of Reynolds stress tensor could be characterized by a set of the eigenvalues and the related eigenvectors. The eigenvalues, called the principal stress σ are defined using Cayley-Hamilton theorem in the form of the characteristic equation:

$$\sigma^3-I\ \sigma^2+II\ \sigma-III=0. \tag{5}$$

The invariants I, II, III are defined as follows:

$$\begin{aligned} I&=b_{kk},\\ II&=-b_{ij}b_{ji}/2,\\ III&=b_{ij}b_{jk}b_{ki}/3=\det\left(b_{ij}\right). \end{aligned} \tag{6}$$

Please note that any anisotropic tensor is traceless from the definition, so the invariant I is identically equal to 0.

3 (An)isotropy rating

From the anisotropy tensor definition it can be seen that none of its eigenvalues can be smaller than −1/3, corresponding to the vanishing of turbulent kinetic energy in that component, nor greater than 2/3, corresponding to the vanishing of other two components (see [2]). This suggests that the range of invariants of the anisotropy tensor is limited by these values. Indeed, Lumley has demonstrated ([2]) that all the possible states of turbulence must be found within the turbulence triangle (so called “Lumley triangle”) in invariant coordinates as shown in Figure 1. The ordinate and the abscissa of this figure are the negative second invariant ($-II$) and the third invariant (III) of the anisotropy tensor, respectively. This graph is called the anisotropy invariant map (AIM).

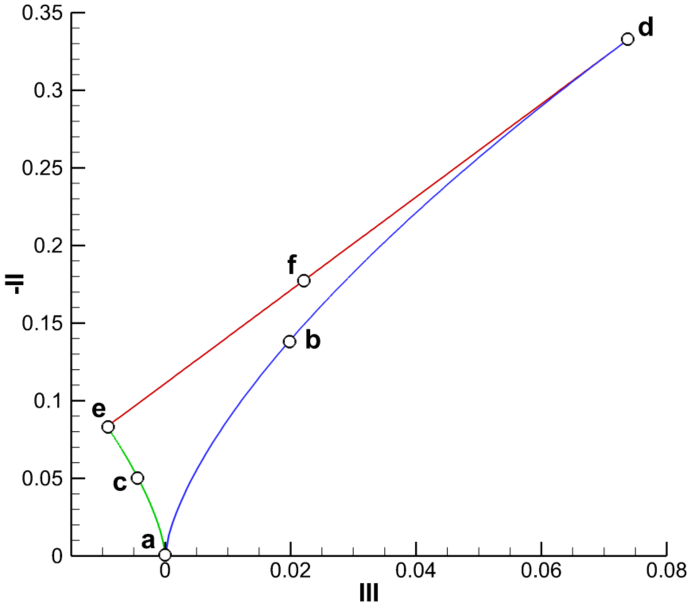


Figure 1: The anisotropy invariant map (cases see Table 1)

The anisotropy invariant map could be subjected to subsequent analysis.

3.1 Anisotropy invariant maps

The origin of the AIM (a): $II = III = 0$ corresponds to three-dimensional (3D) isotropic turbulence.

The two-dimensional (2D) isotropic (i.e. axisymmetric) state of turbulence, where one component of turbulent kinetic energy vanishes with the remaining two being equal, is at the left-hand corner of the triangle (e). The one-dimensional (1D) state of turbulence with only one turbulence component is at the point (d). The isotropy state of the 2 components (axisymmetric) is depicted by green and blue lines. The turbulence along the straight red line connecting the points (e) and (d) is in the 2D state.

The anisotropy invariant map is useful environment in subsequent study of the isotropy development of turbulent flow field. However the shape of the allowed states area is very narrow and the borders are obviously nonlinear. To improve the situation Choi [1] has suggested the modified invariants ξ and η defined by

$$\xi^3 = III/2, \quad \eta^2 = -II/3. \tag{7}$$

The new definition suggests straight lines for axisymmetric cases, however the 2D case is now characterized by nonlinear expression. Situation is shown in Figure 2.

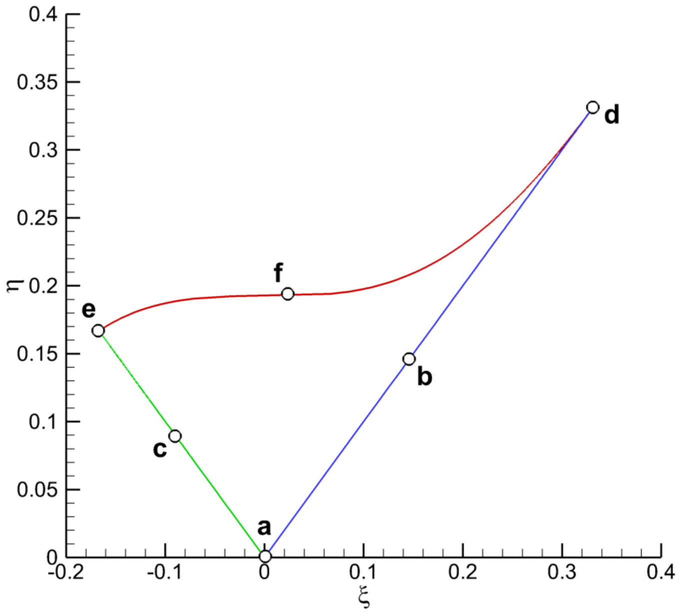


Figure 2: The linearized anisotropy invariant map (cases see Table 1)

The new representation offers much better resolution of the graph close to the 3D isotropy situation.

3.2 Physical interpretation

The energy ellipsoid of homogeneous turbulence has a spherical form.

After an axisymmetric contraction it has a *pancake shape*, since one component of the turbulent kinetic energy is smaller than the other two (green line). In an axisymmetric contraction the turbulence eddies are stretched in the axial direction, making them a rod-like shape.

The energy ellipsoid of turbulence after an axisymmetric expansion has a *cigar shape* since one component of turbulent kinetic energy is greater than the other two (blue line). In this case, however, the turbulence eddies seem to have neither unique structure nor preferred direction as the turbulence is compressed in the axial direction while stretched in the other directions.

The 2D state, when fluctuations in one specific direction vanish, is represented by the flat elliptical shape (red line).

In Table 1 the possible cases are described in details, information adopted from [3]. First of all, the substance of turbulence is characterized. Then, the invariances and corresponding eigenvalues are given. Finally, the shape and geometry of the energy ellipsoid is shown. The cases are equipped with letter labels with reference to Figures 1 and 2.

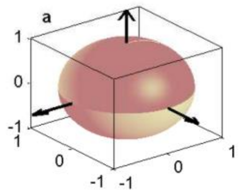
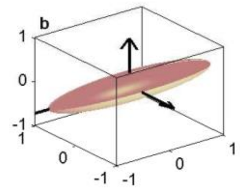
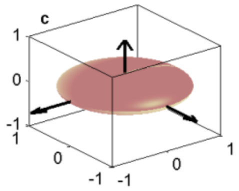
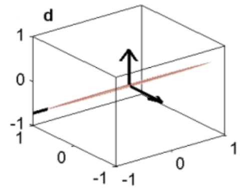
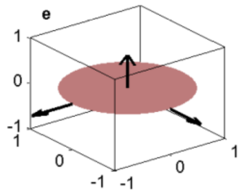
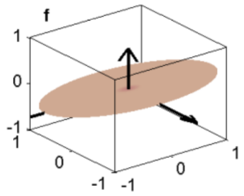
Turbulence state	Invariances	Eigenvalues	Shape	Geometry
a Isotropic	$I = II = III = 0$	$\lambda_i = 0$	Sphere	
b Axisymmetric (cigar shape)	$-II/3 = (III/2)^{2/3}$	$0 < \lambda_3 < 1/3$ $-1/6 < \lambda_2 = \lambda_3 < 0$	Prolate spheroid	
c Axisymmetric (pancake shape)	$-II/3 = (-III/2)^{2/3}$	$-1/3 < \lambda_2 < 0$ $0 < \lambda_1 = \lambda_3 < 1/6$	Oblate spheroid	
d 1D	$III = 2/27$ $II = -1/3$	$\lambda_1 = 2/3$ $\lambda_2 = \lambda_3 = -1/3$	Line	
e 2D, axisymmetric	$III = -1/108$ $II = -1/12$	$\lambda_1 = \lambda_3 = 1/6$ $\lambda_2 = -1/3$	Disk	
f 2D	$-II = 3(1/27 + III)$	$\lambda_1 + \lambda_3 = 1/3$ $\lambda_2 = -1/3$	Ellipsoid	

Table 1 Characteristics of selected situations

To quantify the 3D isotropy of the turbulence Choi [1] has introduced the anisotropic factor F:

$$F = 1 + 27\,III + 9\,II \tag{8}$$

The F vanishes whenever turbulence becomes 2D, and it becomes unity when turbulence enters a 3D isotropic state.

4 Practical example

A practical example of (an)isotropy evaluation will be shown.

The measurement has been carried out using the 3 hot wire probe in grid generated turbulence. The TSI probe is able to indicate simultaneously all 3 components of velocity in the given measuring point. The experiments have been carried out in the ONERA wind tunnel S1 in Modane (France). Cross section was of diameter 8 m, grid generator made from cylindrical rods, mash size about 0.75 m. The velocity was in the case shown here about 25 m/s, position about 7.5 m behind the grid. Velocity components were oriented *u* in the streamwise direction, *v* and *w* in the spanwise directions horizontal and vertical respectively (instead of u_1, u_2 and u_3 used above).

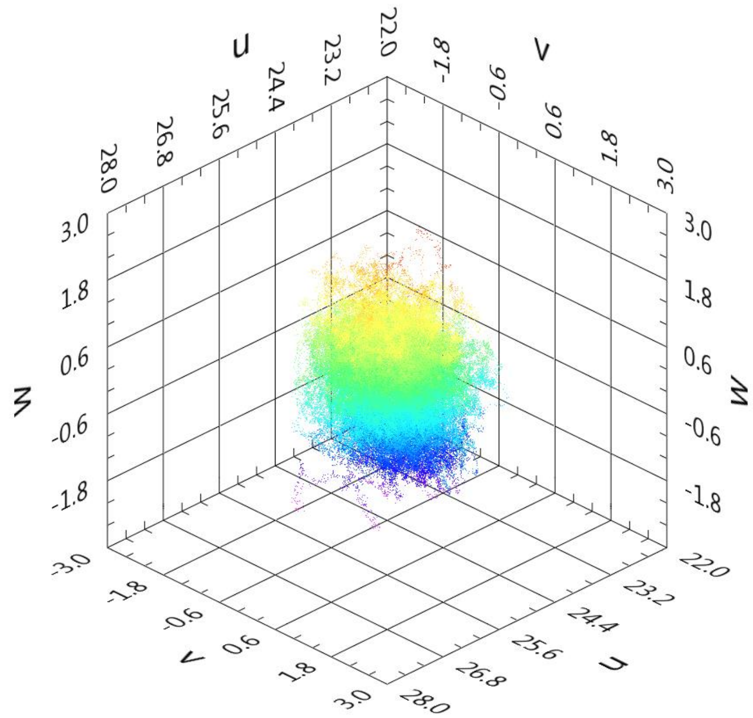


Figure 3: Hodograph of the velocity vector

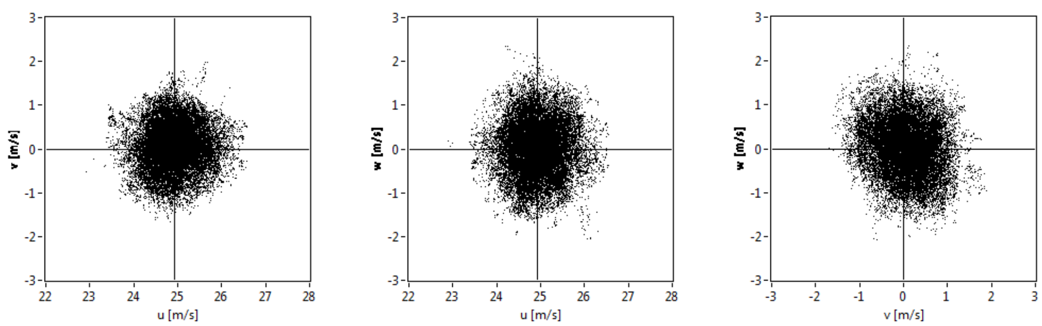


Figure 4: Projections of the hodograph

In Figure 3 there is hodograph of the instantaneous velocity vector in the measuring point during 1 s record (250 thousands values). Colour represents the *w* values. The projections of the points cloud are shown in Figure 4.

The clouds are more or less symmetrical, flow is obviously close to isotropy state.

Now, the covariance matrix of the given case is evaluated:

$$\overline{u_i u_j} = \begin{bmatrix} 1.500 & -0.033 & -0.144 \\ -0.033 & 2.389 & -0.196 \\ -0.144 & -0.196 & 2.345 \end{bmatrix}. \tag{9}$$

The covariance matrix is symmetrical with negative small extradiagonal values. Kinetic energy k and intensity of turbulence were: $k = 3.117$ m^2/s^2, Tu = 5.77 %, the ratios of standard deviations in various directions were evaluated to demonstrate anisotropy: $v/u = 1.262$, $w/u = 1.250$, $v/w = 1.010$. The state could be identified to be close to pancake axisymmetric situation (c) – contraction in streamwise direction.

The isotropic and anisotropic parts of the covariance matrix are as follows:

$$\overline{u_i u_j}^I = \begin{bmatrix} 2.078 & 0 & 0 \\ 0 & 2.078 & 0 \\ 0 & 0 & 2.078 \end{bmatrix}, \tag{10}$$

$$\overline{u_i u_j}^A = \begin{bmatrix} -0.578 & -0.033 & -0.144 \\ -0.033 & 0.311 & -0.196 \\ -0.144 & -0.196 & 0.267 \end{bmatrix}. \tag{11}$$

All the above shown matrices are of physical dimension [m^2/s^2], of course. Now, the nondimensional anisotropic matrix b_{ij} is to be calculated:

$$b_{ij} = \begin{bmatrix} -0.0927 & -0.0053 & -0.0231 \\ -0.0053 & 0.0499 & -0.0314 \\ -0.0231 & -0.0314 & 0.0428 \end{bmatrix}. \tag{12}$$

From this matrix the invariants are determined easily using (6): $I = 0$, $II = -8.019\ 10^{-3}$, $III = -1.420\ 10^{-4}$, $\xi = -0.041$, $\eta = 0.052$.

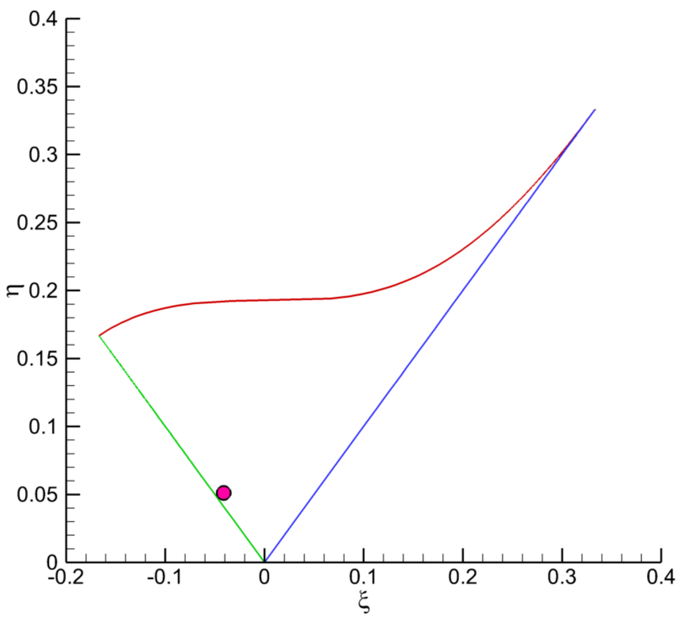


Figure 5: Example case in the linearized anisotropy invariant map

The situation is depicted in the linearized anisotropy invariant map in Figure 5. The red point close to the green line represents the example in question.
In the end, the anisotropic factor is calculated for the case: $F = 0.924$.

5 Conclusions

The analysis method of isotropy or anisotropy properties of a turbulent data obtained using point velocity measurement is shown. Two types of graphical representation of the isotropy invariants are suggested – nonlinear and linearized anisotropy invariant maps.
The representation of isotropy is demonstrated on example of real data from experiments.

References

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