ON THE PROBLEM OF SINGULAR LIMITS IN A MODEL OF RADIATIVE FLOW

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Abstract

We consider a "semi-relativistic" model of radiative viscous compressible Navier-Stokes-Fourier system coupled to the radiative transfer equation extending the classical model introduced in [8] and we study diffusion limits in the case of well-prepared initial data and Dirichlet boundary condition for the velocity field.

Keywords: Radiation hydrodynamics, Navier-Stokes-Fourier system, weak solution, diffusion limits, well-prepared initial data

1 Introduction

In recent works [10] [11] singular limits (low Mach number limit and diffusion limits) for a simplified model of radiation hydrodynamics introduced by Teleaga, Seaïd, Gasser, Klar and Struckmeier in [23] have been presented. A more realistic model was studied in [8] however this more complete model suffers from a non manifestly positive production rate of total entropy, preventing ones from studying these singular limits. Our idea in the present paper is to introduce in the complete model of [8] a perturbed Planck's function and a suitable (relativistic) velocity cut off (this is the meaning we give to "semi-relativistic" model) allowing to recover this crucial positivity property for the production rate of total entropy. As the perturbation will be small (going formally to zero as $c \to \infty$), one can expect to obtain the correct limit regimes.

The motion of the fluid is still described by standard non-relativistic fluid mechanics giving the evolution of the mass density $\varrho = \varrho(t,x)$, the velocity field $\vec{u} = \vec{u}(t,x)$, and the temperature $\vartheta = \vartheta(t,x)$ as functions of the time t and the spatial coordinate $x \in \Omega \subset \mathbb{R}^3$. The effect of radiation is still incorporated in the radiative intensity $I = I(t,x,\vec{\omega},\nu)$, depending on the direction $\vec{\omega} \in \mathcal{S}^2$, where $\mathcal{S}^2 \subset \mathbb{R}^3$ denotes the unit sphere, and the frequency $\nu \geq 0$, but we take into account their relativistic corrections. The evolution of I is described by a transport equation with a source term and the fluid-radiation coupling is expressed through radiative sources in the momentum and energy equations. More precisely, the system of equations to be studied reads as follows:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \vec{u}) = 0 \quad \text{ in } (0,T) \times \Omega, \tag{1}$$

$$\partial_t(\varrho \vec{u}) + \mathrm{div}_x(\varrho \vec{u} \otimes \vec{u}) + \nabla_x p(\varrho,\vartheta) = \mathrm{div}_x \mathbb{S} - \vec{S}_F \quad \text{ in } (0,T) \times \Omega, \tag{2}$$

$$\partial_t \left(\frac{1}{2}\varrho|\vec{u}|^2 + \varrho e(\varrho,\vartheta) \right) + \mathrm{div}_x \left(\left(\frac{1}{2}\varrho|\vec{u}|^2 + \varrho e(\varrho,\vartheta) + p \right) \vec{u} + \vec{q} - \mathbb{S}\vec{u} \right) = -S_E \quad \text{ in } (0,T) \times \Omega, \tag{3}$$

$$\frac{1}{c}\partial_t I + \vec{\omega} \cdot \nabla_x I = S \quad \text{ in } (0,T) \times \Omega \times (0,\infty) \times \mathcal{S}^2. \tag{4}$$

The symbol $p = p(\varrho,\vartheta)$ denotes the thermodynamic pressure and $e = e(\varrho,\vartheta)$ is the specific internal energy, related through Maxwell's equation

$$\frac{\partial e}{\partial \varrho} = \frac{1}{\varrho^2} \left(p(\varrho,\vartheta) - \vartheta \frac{\partial p}{\partial \vartheta} \right). \tag{5}$$

In (2) $\mathbb{S}$ is the viscous stress tensor given by $\mathbb{S} = \mu \left(\nabla_x \vec{u} + \nabla_x^t \vec{u} - \frac{2}{3}\mathrm{div}_x \vec{u} \right) + \eta \, \mathrm{div}_x \vec{u} \, \mathbb{I}$, where the viscosity coefficients $\mu = \mu(\vartheta) > 0$ and $\eta = \eta(\vartheta) \geq 0$ are effective functions of the temperature.

Similarly in (3) $\vec{q}$ is the heat flux given by Fourier's law $\vec{q} = -\kappa\nabla_x\vartheta$, with the heat conductivity coefficient $\kappa = \kappa(\vartheta) > 0$. We suppose that the radiative source S is given by

$$S = \sigma_a\Big[B(\nu,\vec{\omega},\vec{u},\vartheta) - I(t,x,\nu,\vec{\omega})\Big] + \sigma_s\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I(t,x,\nu,\vec{\omega}')\ d\vec{\omega}' - I(t,x,\nu,\vec{\omega})\right) =: S_{a,e} + S_s. \quad (6)$$

In the right-hand side the first term is the emission-absorption contribution where $\sigma_a > 0$ is the absorption coefficient and B is a perturbation of the equilibrium Planck's function given by

$$B(\nu,\vec{\omega},\vec{u},\vartheta) = \frac{2h}{c^2}\frac{\nu^3}{e^{\frac{h\nu}{k\vartheta}\left(1-\alpha\frac{\vec{\omega}\cdot\vec{u}}{c}\right)} - 1}, \quad (7)$$

where h is the Planck's constant, k is the Boltzmann's constant and $0 \le \alpha(\vartheta) \le 1$ is a smooth function, to be determined below. One observes that for $\frac{|\vec{u}|}{c} << 1$ one recovers the standard equilibrium Planck's function $B(\nu,\vartheta) = \frac{2h}{c^2}\frac{\nu^3}{e^{\frac{h\nu}{k\vartheta}} - 1}$.

Note that the idea of this kind of perturbation is not new and has been extensively used in recent works on radiative transfer [4],[5],[7],[6], for exemple in the $M1$ Levermore model [16],[17].

The second term in S is the scattering contribution where $\sigma_s > 0$ is the scattering coefficient and in the right-hand sides of (2) and (3) appear the coupling sources.

$$\vec{S}_F(t,x) = \frac{1}{c}\int_0^\infty\int_{\mathcal{S}^2} \vec{\omega}S\ \mathrm{d}\vec{\omega}\ \mathrm{d}\nu,\ S_E(t,x) = \int_0^\infty\int_{\mathcal{S}^2} S\ \mathrm{d}\vec{\omega}\ \mathrm{d}\nu. \quad (8)$$

We first suppose that the transport coefficients are smooth functions satisfying $\sigma_a(\vartheta,\vec{u}) = \chi(|\vec{u}|)\tilde{\sigma}_a(\vartheta) \ge 0$ and $\sigma_s(\vartheta) \ge 0$ and that both depend neither on angular variable (1 - 4) (isotropy of radiation), nor on frequency (the so called "grey" hypothesis).

The function χ appearing in the emission-absorption coefficient is a C^∞ cut-off satisfying

$$\chi(s) = \begin{cases} 1 & \text{if } s \le c, \\ 0 & \text{if } s \ge c + \beta, \end{cases}$$

for an arbitrary $\beta > 0$. The role of this cut-off is to deal with the singularity of B and its meaning is the following: in the "over-relativistic" regime ($|\vec{u}| \ge c$) where special relativity would be violated, we decide to decouple matter and radiation. Of course this is an arbitrary choice but only a meaningless region with respect to physics is concerned (recall that in the relativistic setting [5], Lorentz factors of the type $\left(1 - \frac{\vec{u}^2}{c^2}\right)^{1/2}$ become singular for $|\vec{u}| = c$). Finally system (1 - 4) is supplemented with the boundary conditions:

$$\vec{u}|_{\partial\Omega} = 0,\ \vec{q}\cdot\vec{n}|_{\partial\Omega} = 0, \quad (9)$$

$$\begin{aligned} &I(t,x,\nu,\vec{\omega})|_{\Gamma_-} = 0, \\ &\Gamma_- \equiv \{\{x,\omega\} \in \partial\Omega \times S^2, \vec{\omega}\cdot\vec{n} \le 0\}, \end{aligned} \quad (10)$$

where $\vec{n}$ denotes the outer normal vector to $\partial\Omega$, and initial conditions

$$(\varrho(t,x),\ \vec{u}(t,x),\ \vartheta(t,x),\ I(t,x,\omega,\nu))|_{t=0} = \left(\varrho^0(x),\ \vec{u}^0(x),\ \vartheta^0(x),\ I^0(x,\vec{\omega},\nu)\right), \quad (11)$$

for any $x \in \Omega$, $\vec{\omega} \in \mathcal{S}^2$,$\nu \in \mathbb{R}_+$.

The relativistic version of system (1 - 10) has been introduced by Pomraning [21] and Mihalas and Weibel-Mihalas [20] and investigated more recently in astrophysics and laser applications (in the inviscid case) by Lowrie, Morel and Hittinger [18] and Buet and Desprès [5], with a special attention to asymptotic regimes, and this last paper was a deep source of inspiration for the present work. Let us mention that a simplified version of the system (non relativistic non conducting fluid at rest) has been investigated by Golse and Perthame in [15] where global existence was proved under very mild hypotheses (transport coefficients may be singular). A global existence result was also proved in [8] for the simplified model (without relativistic corrections), under some cut-off hypotheses on transport coefficients.

Hypotheses:
We consider the pressure in the form

$$p(\varrho,\vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{3}\vartheta^4,\ \ a > 0, \tag{12}$$

where $P : [0,\infty) \to [0,\infty)$ is a given function with the following properties:

$$P \in C^1[0,\infty),\ P(0) = 0,\ P'(Z) > 0,\ \text{for all } Z \geq 0, \tag{13}$$

$$0 < \frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z} < c \text{ for all } Z \geq 0, \tag{14}$$

$$\lim_{Z\to\infty} \frac{P(Z)}{Z^{5/3}} = p_\infty > 0. \tag{15}$$

After Maxwell's equation (5), the specific internal energy e is

$$e(\varrho,\vartheta) = \frac{3}{2}\left(\frac{\vartheta^{5/2}}{\varrho}\right) P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + a\frac{\vartheta^4}{\varrho}, \tag{16}$$

and the associated specific entropy reads

$$s(\varrho,\vartheta) = M\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{4a}{3}\frac{\vartheta^3}{\varrho} \quad \text{with } M'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z^2} < 0. \tag{17}$$

The transport coefficients μ, η, and κ are continuously differentiable functions of the absolute temperature such that

$$0 < c_1(1+\vartheta) \leq \mu(\vartheta),\ \mu'(\vartheta) < c_2,\ 0 \leq \eta(\vartheta) \leq c(1+\vartheta), \tag{18}$$

$$0 < c_1(1+\vartheta^3) \leq \kappa(\vartheta) \leq c_2(1+\vartheta^3) \tag{19}$$

for any $\vartheta \geq 0$. Moreover we assume that σ_a and σ_s are smooth functions such that

$$0 \leq \sigma_a(\vartheta,\vec{u}),\ \sigma_s(\vartheta) \leq c_1,\ \sigma_a(\vartheta,\vec{u})B(\nu,\vec{\omega},\vec{u},\vartheta) \leq c_2, \tag{20}$$

$$\sigma_a(\vartheta,\vec{u})B(\nu,\vec{\omega},\vec{u},\vartheta) \leq h(\nu),\ h \in L^1(0,\infty), \tag{21}$$

where $c_{1,2,3}$ are positive constants. Relations (20 - 21) represent "cut-off" hypotheses neglecting the effect of radiation at large temperature and ultra relativistic velocities (see [22] for physical motivations).

2 Diffusion limits

Diffusion limits consist in supposing that one of the transport coefficient is small while the other is large. These regimes have been introduced by Lowrie, Morel et Hittinger [18] and also considered by Buet and Desprès [5].

In order to identify the appropriate limit regimes we perform two different scalings.

- The first one corresponds to the *equilibrium diffusion regime* defined in [5] by

$$Ma = Sr = Pe = Re = \mathcal{P} = 1,\ \mathcal{C} = \varepsilon^{-1},\ \mathcal{L}_s = \varepsilon^2 \text{ and } \mathcal{L} = \varepsilon^{-1},$$

leading to the primitive system

$$\varepsilon\ \partial_t I + \vec{\omega}\cdot\nabla_x I = \frac{1}{\varepsilon}\sigma_a\left(B - I\right) + \varepsilon\sigma_s\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I\ \mathrm{d}\vec{\omega} - I\right), \tag{22}$$

$$\partial_t\varrho + \mathrm{div}_x(\varrho\vec{u}) = 0, \tag{23}$$

$$\partial_t \left(\varrho\vec{u} + \varepsilon\vec{F}^R\right) + \mathrm{div}_x \left(\varrho\vec{u}\otimes\vec{u} + \mathbb{P}^R\right) + \nabla_x p - \ \mathrm{div}_x\mathbb{S} = 0. \tag{24}$$

$$\partial_t \left(\frac{1}{2}\ \varrho|\vec{u}|^2 + \varrho e + E^R\right) + \mathrm{div}_x \left(\left(\frac{1}{2}\ \varrho|\vec{u}|^2 + \varrho e + p\right)\vec{u} + \frac{\vec{F}^R}{\varepsilon} + \vec{q} - \mathbb{S}\vec{u}\right) = 0, \tag{25}$$

$$\partial_t\left(\varrho s + \varepsilon s_R\right) + \mathrm{div}_x\left(\varrho\vec{u}s + \vec{q}_R\right) + \ \mathrm{div}_x\left(\frac{\vec{q}}{\vartheta}\right) \geq \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x\vec{u} - \frac{\vec{q}\cdot\nabla_x\vartheta}{\vartheta}\right),$$
$$+\frac{1}{\varepsilon}\int_0^\infty\int_{\mathcal{S}^2}\frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1} - \log\frac{n(B)}{n(B)+1}\right]\sigma_a(I-B)\ d\vec{\omega}d\nu$$
$$+\varepsilon\int_0^\infty\int_{\mathcal{S}^2}\frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1} - \log\frac{n(\tilde{I})}{n(\tilde{I})+1}\right]\sigma_s(I-\tilde{I})\ d\vec{\omega}d\nu \tag{26}$$

with

$$\frac{d}{dt}\int_\Omega\left(\varrho\mathcal{E} + E_R\right)\ dx + \frac{1}{\varepsilon}\int_0^\infty\int_{\Gamma_+}\vec{\omega}\cdot\vec{n}\ I\ d\Gamma_+ d\nu = 0, \tag{27}$$

where $\mathcal{E} = \frac{1}{2}\ |\vec{u}|^2 + e$, the entropy of a photon gas is defined as

$$s^R = -\frac{2k}{c^3}\int_0^\infty\int_{\mathcal{S}^2}\nu^2\left[n\log n - (n+1)\log(n+1)\right]d\vec{\omega}d\nu,$$

$n = n(I) = \frac{c^2 I}{2h\alpha^3\nu^3}$ is the occupation number, the radiative entropy flux is defined as

$$\vec{q}^R = -\frac{2k}{c^2}\int_0^\infty\int_{\mathcal{S}^2}\nu^2\left[n\log n - (n+1)\log(n+1)\right]\vec{\omega}\ d\vec{\omega}d\nu,$$

$\log\frac{n(B)}{n(B)+1} = -\frac{h\nu}{k\vartheta}\left(1 - \alpha\frac{\vec{\omega}\cdot\vec{u}}{c}\right)$, where B is the Planck's function. The radiative energy E^R is defined as

$$E^R(t,x) = \frac{1}{c}\int_{\mathcal{S}^2}\int_0^\infty I(t,x,\vec{\omega},\nu)\ \mathrm{d}\vec{\omega}\ \mathrm{d}\nu, \tag{28}$$

and the radiative momentum

$$\vec{F}^R(t,x) = \int_{\mathcal{S}^2}\int_0^\infty \vec{\omega}I(t,x,\vec{\omega},\nu)\ \mathrm{d}\vec{\omega}\ \mathrm{d}\nu. \tag{29}$$

- The second one is the " non-equilibrium diffusion regime" also defined in [5] by

$$Ma = Sr = Pe = Re = \mathcal{P} = 1,\ \mathcal{C} = \varepsilon^{-1},\ \mathcal{L} = \varepsilon^2 \text{ and } \mathcal{L}_s = \varepsilon^{-1}.$$

One checks that equations (23) (24) (25) and (27) still hold in this scaling. The new transport equation is

$$\varepsilon\ \partial_t I + \vec{\omega}\cdot\nabla_x I = \varepsilon\sigma_a\left(B - I\right) + \frac{1}{\varepsilon}\sigma_s\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I\ \mathrm{d}\vec{\omega} - I\right), \tag{30}$$

and the new entropy inequality is

$$\partial_t\left(\varrho s + \varepsilon s_R\right) + \mathrm{div}_x\left(\varrho\vec{u}s + \vec{q}_R\right) + \mathrm{div}_x\left(\frac{\vec{q}}{\vartheta}\right) \geq \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x\vec{u} - \frac{\vec{q}\cdot\nabla_x\vartheta}{\vartheta}\right)$$
$$+\varepsilon\int_0^\infty\int_{\mathcal{S}^2}\frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1} - \log\frac{n(B)}{n(B)+1}\right]\sigma_a(I-B)\ d\vec{\omega}d\nu$$
$$+\frac{1}{\varepsilon}\int_0^\infty\int_{\mathcal{S}^2}\frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1} - \log\frac{n(\tilde{I})}{n(\tilde{I})+1}\right]\sigma_s(I-\tilde{I})\ d\vec{\omega}d\nu. \tag{31}$$

Target system in the equilibrium-diffusion regime

$$\partial_t\varrho + \mathrm{div}_x(\varrho\vec{u}) = 0, \tag{32}$$

$$\partial_t(\varrho\vec{u}) + \mathrm{div}_x(\varrho\vec{u}\otimes\vec{u}) + \nabla_x\mathbf{p} = \mathrm{div}_x\mathbb{S}, \tag{33}$$

$$\partial_t\left(\frac{1}{2}\,\varrho|\vec{u}_0|^2 + \varrho\mathbf{e}\right) + \mathrm{div}_x\left(\left(\frac{1}{2}\,\varrho|\vec{u}|^2 + \varrho\mathbf{e} + \mathbf{p}\right)\vec{u} + \vec{\mathbf{q}} - \mathbb{S}\vec{u}\right) = 0, \tag{34}$$

$$\partial_t\left(\varrho\mathbf{s}\right) + \mathrm{div}_x\left(\varrho\mathbf{s}\vec{u}\right) + \mathrm{div}_x\left(\frac{\vec{\mathbf{q}}}{\vartheta}\right) = \frac{1}{\vartheta}\left(\mathbb{S}:\nabla_x\vec{u} - \frac{\vec{\mathbf{q}}\cdot\nabla_x\vartheta}{\vartheta}\right), \tag{35}$$

$$I = B(\nu,\vartheta), \tag{36}$$

where $\mathbf{p}(\varrho,\vartheta) = p(\varrho,\vartheta) + \frac{a}{3}\vartheta^4$, $\mathbf{e}(\varrho,\vartheta) = e(\varrho,\vartheta) + \frac{a}{\varrho}\vartheta^4$, $\mathbf{k}(\vartheta_0) = \kappa(\vartheta) + \frac{4a}{3\sigma_a}\vartheta^3$, $\vec{\mathbf{q}} = -\mathbf{k}(\vartheta)\nabla_x\vartheta$ and $\varrho\mathbf{s} = \varrho s + \frac{4}{3}a\vartheta^3$.

We also get boundary conditions

$$\vec{u}|_{\partial\Omega} = 0,\ \nabla\vartheta\cdot\vec{n}|_{\partial\Omega} = 0, \tag{37}$$

and initial conditions

$$\left(\varrho(x,t),\ \vec{u}(x,t),\ \vartheta(x,t)\right)|_{t=0} = \left(\varrho^0(x),\ \vec{u}^0(x),\ \vartheta^0(x)\right), \tag{38}$$

for any $x\in\Omega$ with the following compatibility conditions

$$\vec{u}^0(x)|_{\partial\Omega} = 0,\ \nabla\vartheta^0\cdot\vec{n}|_{\partial\Omega} = 0. \tag{39}$$

As expected, this system corresponds to a viscous compressible heat-conductive fluid at local thermodynamical equilibrium with radiation, equilibrium being achieved between matter and radiation with radiative intensity $I = B(\nu,\vartheta)$, corresponding to the black-body radiation at temperature ϑ with radiative energy $E_R(\vartheta) = a\vartheta^4$.

From the classical results of Matsumura and Nishida [19] it can be derived the existence results see [12].

We adapt from [13] the necessary definitions to the formalism of essential and residual sets. Given three numbers $\overline{\varrho}\in\mathbb{R}_+$, $\overline{\vartheta}\in\mathbb{R}_+$ and $\overline{E}\in\mathbb{R}_+$ we define $\mathcal{O}^H_{ess}$ the set of hydrodynamical essential values

$$\mathcal{O}^H_{ess} := \left\{(\varrho,\vartheta)\in\mathbb{R}^2\ :\ \frac{\overline{\varrho}}{2} < \varrho < 2\overline{\varrho},\ \frac{\overline{\vartheta}}{2} < \vartheta < 2\overline{\vartheta}\right\}, \tag{40}$$

and $\mathcal{O}^R_{ess}$ the set of radiative essential values

$$\mathcal{O}^R_{ess} := \left\{E^R\in\mathbb{R}\ :\ \frac{\overline{E}}{2} < E^R < 2\overline{E}\right\}, \tag{41}$$

with $\mathcal{O}_{ess} := \mathcal{O}^H_{ess}\cup\mathcal{O}^R_{ess}$, and their residual counterparts

$$\mathcal{O}^H_{res} := (\mathbb{R}_+)^2\backslash\mathcal{O}^H_{ess},\quad \mathcal{O}^R_{res} := \mathbb{R}_+\backslash\mathcal{O}^R_{ess},\quad \mathcal{O}_{res} := (\mathbb{R}_+)^3\backslash\mathcal{O}_{ess}. \tag{42}$$

Theorem 2.1 *Let $\Omega\subset\mathbb{R}^3$ be a bounded domain of class $C^{2,\nu}$. Assume that the thermodynamic functions p, e, s satisfy hypotheses (12 - 17) with $P\in C^1[0,\infty)\cap C^2(0,\infty)$, and that the transport coefficients μ, η, κ, σ_a, σ_s and the equilibrium function B comply with (18) - (21).*

Let $(\varrho_\varepsilon,\vec{u}_\varepsilon,\vartheta_\varepsilon,I_\varepsilon)$ be a weak solution to the scaled radiative Navier-Stokes system (22 - 27) for $(t,x,\vec{\omega},\nu)\in[0,T]\times\Omega\times\mathcal{S}^2\times\mathbb{R}_+$, supplemented with boundary conditions (9 - 10) and initial conditions $(\varrho_{0,\varepsilon},\vec{u}_{0,\varepsilon},\vartheta_{0,\varepsilon},I_{0,\varepsilon})$ such that

$$\varrho_\varepsilon(0,\cdot) = \varrho_0 + \sqrt{\varepsilon}\varrho^{(1)}_{0,\varepsilon},\quad \vec{u}_\varepsilon(0,\cdot) = \vec{u}_{0,\varepsilon},\quad \vartheta_\varepsilon(0,\cdot) = \vartheta_0 + \sqrt{\varepsilon}\vartheta^{(1)}_{0,\varepsilon},$$

where $(\varrho_0,\vec{u},\vartheta_0)\in H^3(\Omega)$ are smooth functions such that (ϱ_0,ϑ_0) belong to the set $\mathcal{O}^H_{ess}$ where $\overline{\varrho}>0$, $\overline{\vartheta}>0$, are two constants and $\int_\Omega\varrho^{(1)}_{0,\varepsilon}\,dx = 0$, $\int_\Omega\vartheta^{(1)}_{0,\varepsilon}\,dx = 0$. Suppose also that

$$\vec{u}_{0,\varepsilon}\to\vec{u}_0\ \ \textit{strongly in } L^\infty(\Omega;\mathbb{R}^3),\ \varrho^{(1)}_{0,\varepsilon}\to\varrho^{(1)}_0\ \textit{strongly in } L^2(\Omega),$$

$$\vartheta^{(1)}_{0,\varepsilon}\to\vartheta^{(1)}_0\ \textit{strongly in } L^2(\Omega).$$

Then up to subsequences

$$\varrho_\varepsilon \to \varrho \ \ \textit{strongly in } L^\infty(0,T;L^{\frac{5}{3}}(\Omega)),\ \vec{u}_\varepsilon \to \vec{u} \textit{ strongly in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)),$$

$$\vartheta_\varepsilon \to \vartheta \ \ \textit{strongly in } L^\infty(0,T;L^4(\Omega)),\ I_\varepsilon \to B(\nu,\theta) \ \ \textit{strongly in } L^\infty((0,T)\times\Omega\times\mathcal{S}^2)\times(0,\infty)),$$

where $(\varrho,\vec{u},\vartheta)$ *is the smooth solution of the equilibrium decoupled system (32)-(34) on* $[0,T]\times\Omega$, *with initial data* $(\varrho_0,\vec{u}_0,\vartheta_0)$.

The target system in the non-equilibrium diffusion regime

We obtain a compressible Navier-Stokes-Fourier system with sources coupled to a diffusion equation for N.

$$\partial_t\varrho + \mathrm{div}_x(\varrho\vec{u}) = 0, \tag{43}$$

$$\partial_t(\varrho\vec{u}) + \mathrm{div}_x(\varrho\vec{u}\otimes\vec{u}) + \nabla_x\mathbf{p} = \mathrm{div}_x\mathbb{S}, \tag{44}$$

$$\partial_t\left(\frac{1}{2}\ \varrho|\vec{u}_0|^2 + \varrho\mathbf{e}\right) + \mathrm{div}_x\left(\left(\frac{1}{2}\ \varrho|\vec{u}|^2 + \varrho\mathbf{e} + \mathbf{p}\right)\vec{u} + \vec{\mathbf{q}} - \mathbb{S}\vec{u}\right) = 0, \tag{45}$$

$$\partial_t\left(\varrho s\right) + \mathrm{div}_x\left(\varrho s\vec{u}\right) + \mathrm{div}_x\left(\frac{\vec{q}}{\vartheta}\right) = \frac{1}{\vartheta}\left(\mathbb{S}:\nabla_x\vec{u} - \frac{\vec{q}\cdot\nabla_x\vartheta}{\vartheta}\right) + \frac{1}{3}\frac{\nabla_x N\cdot\vec{u}}{\vartheta} - \frac{\sigma_a(\vartheta)}{\vartheta}\left(a\vartheta^4 - N\right), \tag{46}$$

$$\partial_t N - \frac{1}{3}\ \mathrm{div}_x\left(\frac{1}{\sigma_s(\vartheta)}\nabla_x N\right) = \sigma_a(\vartheta)\left(a\vartheta^4 - N\right), \tag{47}$$

where $\mathbf{p} = p + \frac{1}{3}N$, $\mathbf{e} = e + \frac{N}{\varrho}$ and $\vec{\mathbf{q}} = \kappa\nabla_x\vartheta + \frac{1}{3\sigma_s}\nabla_x N$ with boundary conditions

$$\vec{u}|_{\partial\Omega} = 0,\ \nabla\vartheta\cdot\vec{n}|_{\partial\Omega} = 0,\ N|_{\partial\Omega} = 0, \tag{48}$$

initial conditions

$$\left(\varrho(x,t),\ \vec{u}(x,t),\ \vartheta(x,t),N(x,t)\right)|_{t=0} = \left(\varrho^0(x),\ \vec{u}^0(x),\ \vartheta^0(x),N^0(x)\right), \tag{49}$$

for any $x\in\Omega$, with $N^0(x) = \int_0^\infty\int_{\mathcal{S}^2} I^0(x,\nu,\vec{\omega})\ d\vec{\omega}\ d\nu$ and the compatibility conditions

$$\vec{u}^2|_{\partial\Omega} = 0,\ \nabla\vartheta^0\cdot\vec{n}|_{\partial\Omega} = 0,\ N^0\big|_{\partial\Omega} = 0. \tag{50}$$

It will be useful as in [5] to define the non-equilibrium temperature θ_r by

$$N = a\theta_r^4. \tag{51}$$

In analogy with previous works on asymptotic analysis of radiative transfer equation (see [2], [3]) we call (43)-(49) the Navier-Stokes-Rosseland system. As in the equilibrium case, we have a global existence result for solutions of this problem for small data for more details see [12].

Theorem 2.2 *Let* $\Omega\subset\mathbb{R}^3$ *be a bounded domain of class* $C^{2,\nu}$. *Assume that the thermodynamic functions* p, e, s *satisfy hypotheses (12 - 17) with* $P\in C^1[0,\infty)\cap C^2(0,\infty)$, *and that the transport coefficients* μ, λ, κ, σ_a, σ_s *and the equilibrium function* B *comply with (18) - (21).*

Let $(\varrho_\varepsilon,\vec{u}_\varepsilon,\vartheta_\varepsilon,I_\varepsilon)$ *be a weak solution to the system (23) (24) (25), (27), (30), (31) for* $(t,x,\vec{\omega},\nu)\in[0,T]\times\Omega\times\mathcal{S}^2\times\mathbb{R}_+$, *supplemented with the boundary conditions (9 - 10) and the initial conditions* $(\varrho_{0,\varepsilon},\vec{u}_{0,\varepsilon},\vartheta_{0,\varepsilon},I_{0,\varepsilon})$ *such that*

$$\varrho_\varepsilon(0,\cdot) = \varrho_0 + \sqrt{\varepsilon}\varrho_{0,\varepsilon}^{(1)},\quad \vec{u}_\varepsilon(0,\cdot) = \vec{u}_{0,\varepsilon},\quad \vartheta_\varepsilon(0,\cdot) = \vartheta_0 + \sqrt{\varepsilon}\vartheta_{0,\varepsilon}^{(1)},\quad I_\varepsilon(0,\cdot) = I_0 + \sqrt{\varepsilon}I_{0,\varepsilon}^{(1)},$$

where the functions $(\varrho_0,\vec{u},\vartheta_0)$ *and* $x\to I_0(x,\vec{\omega},\nu)$ *belong to* $H^3(\Omega)$ *and are such that* $(\varrho_0,\vartheta_0,E_R(I_0))$ *belong to the set* $\mathcal{O}_{ess}$. *Suppose also that*

$$\vec{u}_{0,\varepsilon}\to\vec{u}_0 \ \ \textit{strongly in } L^\infty(\Omega;\mathbb{R}^3),\ \varrho_{0,\varepsilon}^{(1)}\to\varrho_0^{(1)} \textit{ strongly in } L^2(\Omega),$$

$$\vartheta_{0,\varepsilon}^{(1)}\to\vartheta_0^{(1)} \textit{ strongly in } L^2(\Omega),\ I_{0,\varepsilon}^{(1)}\to I_0^{(1)} \textit{ strongly in } L^\infty((0,T)\times\Omega\times(0,\infty)).$$

Then up to subsequences

$$\varrho_\varepsilon \to \varrho \ \text{ strongly in } L^\infty(0,T;L^{\frac{5}{3}}(\Omega)),\ \vec{u}_\varepsilon \to \vec{u} \text{ strongly in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)),$$

$$\vartheta_\varepsilon \to \vartheta \ \text{ strongly in } L^\infty(0,T;L^4(\Omega)),\ N_\varepsilon \to N \ \text{ strongly in } L^\infty((0,T)\times\Omega),$$

where $N_\varepsilon = \int_0^\infty \int_{\mathcal{S}^2} I_\varepsilon \, d\vec{\omega} \ d\nu$ *and* $(\varrho, \vec{u}, \vartheta, N)$ *is the smooth solution of the Navier-Stokes-Rosseland system (43)-(47) on* $[0,T]\times\Omega$ *with initial data* $(\varrho_0, \vec{u}_0, \vartheta_0, N_0)$.

Proofs of Theorems 2.1, 2.2 are based on the theory of singular limits [13] and the relative entropy inequality [14]. We just give the sketch of the proof. We introduce a *relative entropy inequality* satisfied by any weak solution $(\varrho, \vec{u}, \vartheta, I)$ of the radiative Navier-Stokes -Fourier system.

Let us consider a set $\{r, \Theta, \vec{U}\}$ of arbitrary smooth functions such that r and Θ are bounded below away from zero and $\vec{U}\Big|_{\partial\Omega} = 0$. We call *ballistic free energy* the thermodynamical potential given by $H_\Theta(\varrho,\vartheta) = \varrho e(\varrho,\vartheta) - \Theta \varrho s(\varrho,\vartheta)$, and *radiative ballistic free energy* the potential $H_\Theta^R(I) = E^R(I) - \Theta s^R(I)$. The *relative entropy* is then defined by

$$\mathcal{E}(\varrho,\vartheta | r,\Theta) := H_\Theta(\varrho,\vartheta) - \partial_\rho H_\Theta(r,\Theta)(\varrho - r) - H_\Theta(r,\Theta).$$

Then the relative entropy inequality of the radiative Navier-Stokes-Fourier system is the following

$$\begin{aligned}
&\int_\Omega \left(\frac{1}{2}\ \varrho_\varepsilon |\vec{u}_\varepsilon - \vec{U}|^2 + \mathcal{E}\left(\varrho_\varepsilon, \vartheta_\varepsilon | r, \Theta\right) + \varepsilon H^R(I_\varepsilon) \right)(\tau,\cdot)\ dx + \int_0^\tau \int_{\Gamma_+} \vec{\omega}\cdot\vec{n}_x I_\varepsilon(t,x,\vec{\omega},\nu)\ d\Gamma\ d\nu\ dt \\
&\quad + \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta_\varepsilon}\left(\mathbb{S}_\varepsilon : \nabla_x \vec{u}_\varepsilon - \frac{\vec{q}_\varepsilon \cdot \nabla_x \vartheta_\varepsilon}{\vartheta_\varepsilon} \right)\ dx\ dt \\
&\quad + \int_0^\tau \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \frac{\Theta}{\nu}\left[\log \frac{n(I_\varepsilon)}{n(I_\varepsilon)+1} - \log \frac{n(B_\varepsilon)}{n(B_\varepsilon)+1} \right] \sigma_{a_\varepsilon^{(j)}} (B_\varepsilon - I_\varepsilon)\ d\vec{\omega}\ d\nu\ dx\ dt \\
&\quad + \int_0^\tau \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \frac{\Theta}{\nu}\left[\log \frac{n(I_\varepsilon)}{n(I_\varepsilon)+1} - \log \frac{n(\tilde{I}_\varepsilon)}{n(\tilde{I}_\varepsilon)+1} \right] \sigma_{s_\varepsilon^{(j)}} (\tilde{I}_\varepsilon - I_\varepsilon)\ d\vec{\omega}\ d\nu\ dx\ dt, \\
&\quad \le \int_\Omega \frac{1}{2}\left(\varrho_{0,\varepsilon} |\vec{u}_{0,\varepsilon} - \vec{U}(0,\cdot)|^2 + \mathcal{E}\left(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon} | r(0,\cdot), \Theta(0,\cdot)\right) + \varepsilon H^R(I_{0,\varepsilon}) \right)\ dx \\
&+ \int_0^\tau \int_\Omega \varrho_\varepsilon (\vec{u}_\varepsilon - \vec{U}) \cdot \nabla_x \vec{U} \cdot \left(\vec{U} - \vec{u}_\varepsilon \right)\ dx\ dt + \int_0^\tau \int_\Omega \varrho_\varepsilon \left(s_\varepsilon - s(r,\Theta) \right) \left(\vec{U} - \vec{u}_\varepsilon \right) \cdot \nabla_x \Theta\ dx\ dt \\
&\quad + \int_0^\tau \int_\Omega \left(\varrho_\varepsilon \left(\partial_t \vec{U} + \vec{U}\cdot\nabla_x \vec{U} \right) \cdot \left(\vec{U} - \vec{u}_\varepsilon \right) \right)\ dx\ dt \\
&\quad - \int_0^\tau \int_\Omega \left(p_\varepsilon \ \mathrm{div}_x \vec{U} - \mathbb{S}_\varepsilon : \nabla_x \vec{U} \right)\ dx\ dt - \int_0^\tau \int_\Omega \left(\varepsilon s_\varepsilon^R \partial_t \Theta + \vec{q}_\varepsilon^R \cdot \nabla_x \Theta \right)\ dx\ dt \\
&\quad - \int_0^\tau \int_\Omega \left(\varrho_\varepsilon \left(s_\varepsilon - s(r,\Theta) \right) \partial_t \Theta \right)\ dx\ dt - \int_0^\tau \int_\Omega \varrho_\varepsilon \left(s_\varepsilon - s(r,\Theta) \right) \vec{U}\cdot\nabla_x\Theta\ dx\ dt \\
&\quad - \int_0^\tau \int_\Omega \frac{\vec{q}_\varepsilon}{\vartheta_\varepsilon} \cdot \nabla_x \Theta\ dx\ dt + \int_0^\tau \int_\Omega \left(\left(1 - \frac{\varrho_\varepsilon}{r} \right) \partial_t p(r,\Theta) - \frac{\varrho_\varepsilon}{r}\ \vec{u}_\varepsilon \nabla_x p(r,\Theta) \right)\ dx\ dt \\
&\quad - \int_0^\tau \int_\Omega \left(\varepsilon \vec{F}_\varepsilon^R \cdot \partial_t \vec{U} + \mathbb{P}_\varepsilon^R : \nabla_x \vec{U} \right)\ dx\ dt. \qquad (52)
\end{aligned}$$

For more details see [12].

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