STABILITY OF FINITE ELEMENT – FINITE VOLUME DISCRETIZATIONS OF CONVECTION-DIFFUSION-REACTION EQUATIONS

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Abstract

We consider a time-dependent and a steady linear convection-diffusion-reaction equation. These equations are approximately solved by a combined finite element – finite volume method: the diffusion term is discretized by Crouzeix-Raviart piecewise linear finite elements on a triangular grid, and the convection and reaction term by upwind barycentric finite volumes. In the unsteady case, the implicit Euler method is used as time discretization. This scheme is unconditionally L^2-stable, uniformly with respect to the diffusion coefficient.

Keywords: convection-diffusion-reaction equation, combined finite element - finite volume method, Crouzeix-Raviart finite elements, barycentric finite volumes, upwind method, stability.

1 Introduction

Consider the convection-diffusion-reaction equation

$$\partial_t u - \nu\,\Delta u + \nabla\cdot(u\beta) + \mu\, u = g \quad \text{in } \ \Omega\times(0,T), \tag{1}$$

with the initial and boundary conditions

$$u(x,0) = u^{(0)}(x) \ \text{ for } \ x\in\Omega, \qquad u\,|\,\partial\Omega\times(0,T) = 0, \tag{2}$$

respectively. Here $\Omega\subset\mathbb{R}^2$ is a bounded open polygon with Lipschitz boundary, ν and T are positive reals, and $\beta:\Omega\mapsto\mathbb{R}^2,\ \mu,\ u^{(0)}:\Omega\mapsto\mathbb{R}$ as well as $g:\Omega\times(0,T)\mapsto\mathbb{R}$ are given functions. Our key assumptions concern the advective velocity β: we require that $\beta\in H^1(\Omega)^2,\ \nabla\cdot\beta\in L^p(\Omega)$ for some $p\in(2,\infty]$ and

$$\nabla\cdot\beta/2 + \mu \geq 0, \quad -\beta\cdot\nabla\varphi \geq \underline{\beta} \quad \text{in } \ \Omega \tag{3}$$

for some function $\varphi\in C^1(\overline{\Omega})$ and some constant $\underline{\beta}>0$. In the case that β is constant, a suitable function φ is given by $\varphi(x) := -\beta\cdot x$.

We also consider the steady variant of problem (1), (2), that is,

$$-\nu\,\Delta U + \nabla\cdot(U\beta) + \mu U = G \quad \text{in } \ \Omega, \tag{4}$$

$$U|\partial\Omega = 0, \tag{5}$$

where $G:\Omega\mapsto\mathbb{R}$ is another given function. These problems are of particular interest in the convection-dominated regime, that is, if $\nu\ll|\beta|$, an interest that seems to be motivated by the belief that the preceding problems in the convection-dominated case show some affinity (although distant) with the Navier-Stokes system in the same regime. In this spirit, numerical schemes working well for that latter system are sometimes reduced to problem (1), (2) or (4), (5) so that they may be accessible to theoretical studies regarding stability or accuracy.

In the work at hand, we consider a discretization of (1), (2) and (4), (5), respectively, that is motivated in this way. This scheme may be described as follows: the diffusion term in (1) and (4) is discretized by piecewise linear Crouzeix-Raviart finite elements, and the convective term by an upwind finite volume method based on barycentric finite volumes on a triangular grid. Choosing an explicit time discretization, Feistauer e.a. [1, Section 7], [2, Chapter 4.4] tested this FE-FV method in the case of high-speed compressible Navier-Stokes flows in complex geometries and obtained very satisfactory results.

In [3], we applied this FE-FV method to problem (1), (2), using the implicit Euler method as time discretization. Under the assumption that $\nabla \cdot \beta = 0$, $\mu = 0$ instead of $(3)_1$, we showed for a shape-regular grid (minimum angle condition) that the approximate solution provided by this approach may be estimated in the L^2-norm against the data, with the constant in this estimate being independent of the diffusion parameter ν, and depending polynomially on $\underline{\beta}^{-1}$, $\|\beta\|_{1,p}$ and on an upper bound of φ and $\nabla\varphi$. An analogous result was established with respect to problem (4), (5).

The fact that in the work at hand, we only suppose $(3)_1$ instead of requiring $\nabla \cdot \beta = 0$ and $\mu = 0$ might be expected to present no major additional difficulty. However, it turned out that stronger assumptions have to be imposed. In fact, we introduce the additional condition $\nabla \cdot \beta \in L^p(\Omega)$ for some $p > 2$, and we require that the maximal diameter of the triangles of the grid has to be small with respect to $\underline{\beta}$, $\|\nabla \cdot \beta\|_p$ and an upper bound on φ and $\nabla\varphi$.

We remark that our approach also carries through if the functions β, μ and φ in (3) depend on time, provided that the spatial H^1-norm of β as well as the spatial L^p-norm of $\nabla \cdot \beta$ is bounded uniformly with respect to time, and (3) holds at any time with the same constant $\underline{\beta}$. Moreover the diffusion term $\nu\,\Delta u$ may be replaced by an elliptic operator in divergence form $\nabla \cdot (A \cdot \nabla u)$, where $A = A(x)$ or $A = A(x, t)$ is a symmetric matrix in $\mathbb{R}^{2\times 2}$ which is positive definite uniformly with respect to x and (in the unsteady case) t. Our theory should be expected to hold in the 3D case as well. There is only one point that might need some effort, that is, to prove an analogue or find a replacement of equation (9) pertaining to the discrete L^2-scalar product.

There is an extensive literature on stability estimates of various discretizations of problem (1), (2) and (4), (5). In [3], we tried to present a survey on this subject, listing a considerable number of references. Here we only remark that stability estimates previous to [3] either involve constants depending exponentially on ν or on quantities related to β, or they depend on T (see [4] for example), or condition $(3)_1$ is replaced by the stronger assumption $\nabla \cdot \beta/2 + \mu \geq \delta$ for some $\delta > 0$. Among those previous articles, the one closest related to [3] and to the work at hand is reference [5], which deals with a class of discontinuous Galerkin discretizations of (4), (5). These discretizations lead to a discrete convection term similar to ours. Moreover – and more importantly –, to our knowledge reference [5] is the only one previous to [3] which is bases its theory on condition $(3)_2$.

This condition means that the advective velocity β exhibits neither closed curves nor stationary points (points $x \in \Omega$ with $\beta(x) = 0$). In fact, it is shown in [6] that if β is smooth and presents these geometrical properties, there is a function φ with $(3)_2$. This result was generalized to the case $\beta \in W^{1,\infty}(\Omega)^2$ in [5]. According to [3, Section 5], assumption $(3)_2$ is in a certain sense necessary and sufficient for stability estimates with constants independent of ν.

2 Notation. FE-FV discretization of (1), (2) and (4), (5), respectively. Statement of main results.

As already indicated in Section 1, we assume that $\Omega \subset \mathbb{R}^2$ is a bounded open polygon with Lipschitz boundary. The functions β, φ, g, u_0 and G and the constants ν, T, $\underline{\beta} \in (0, \infty)$ were also introduced in Section 1, with $\beta \in H^1(\Omega)^2$, $\nabla \cdot \beta \in L^p(\Omega)^2$ for some $p \in (2, \infty]$, $\varphi \in C^1(\overline{\Omega})$ and μ being such that (3) is satisfied. We assume that $\mu \in L^1(\Omega)$, $g \in C^0\big([0,T], H^1(\Omega)\big)$ and $u^{(0)}, G \in H^1(\Omega)$. The functions $g(\,\cdot\,, t)$ and G are required to belong to $H^1(\Omega)$ instead of only to $L^2(\Omega)$ so that they admit traces on edges, in view of an interpolation operator we will introduce below.

By adding a constant to φ, we may suppose without loss of generality that $\varphi(x) \geq \varphi_0$ $(x \in \overline{\Omega})$ for some $\varphi_0 > 0$. For example, in the case $\beta = \beta_0$ for some $\beta_0 \in \mathbb{R}^2\backslash\{0\}$, we may put $\varphi(x) := 2|\beta_0|\,diam(\Omega) - \beta_0 \cdot (x - x_0)$, where x_0 is an arbitrary but fixed point in Ω. Obviously there is a constant $\varphi_1 > 0$ with $\varphi(x) \leq \varphi_1$ and $|\nabla\varphi(x)| \leq \varphi_1$ for $x \in \overline{\Omega}$. We further introduce a parameter $\sigma_0 \in (0, 1)$, which will appear in condition (6) below. The set Ω, the functions β, φ, μ, g, u_0 and G as well as the numbers ν, T, φ_0, φ_1, $\underline{\beta}$ and σ_0 will be kept fixed throughout.

By the symbol $\mathfrak{C}$, we denote constants that may depend on σ_0, $\operatorname{diam}\Omega$, $\underline{\beta}$, φ_0 and φ_1, with polynomial dependence on $diam\,\Omega$, $\underline{\beta}^{-1}$, φ_0 and φ_1. This last feature is important because we want to control how our estimates are influenced by β. This influence not only manifests itself by the factor $1 + \|\beta\|_{1,2}$ appearing in Theorem 2.1 and 2.2, but also via the quantities $\underline{\beta}$, φ_0 and φ_1.

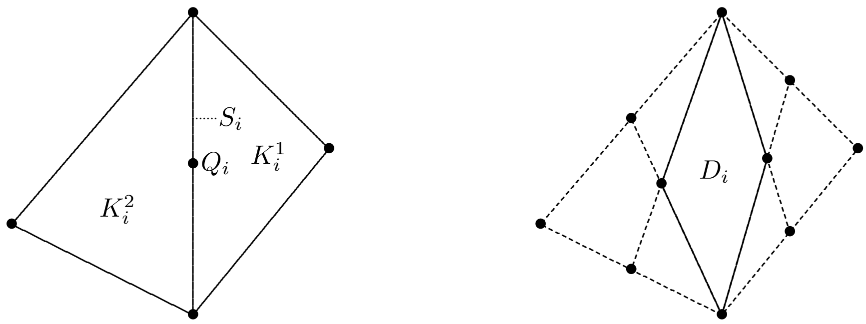

Figure 1: A quadrangle $\overline{K_i^1} \cup \overline{K_i^2}$ (left) and the quadrangle D_i inside $\overline{K_i^1} \cup \overline{K_i^2}$ (right).

We consider triangulations $\mathfrak{T}$ of Ω with the following three properties: Firstly, $\mathfrak{T}$ is a finite set of open triangles $K \subset \mathbb{R}^2$ with $\overline{\Omega} = \cup\{\overline{K} \,:\, K \in \mathfrak{T}\}$. Secondly, if $K_1, K_2 \in \mathfrak{T}$ with $\overline{K_1} \cap \overline{K_2} \neq \emptyset$ and $K_1 \neq K_2$, then $\overline{K_1} \cap \overline{K_2}$ is a common vertex or a common side of K_1 and K_2. And thirdly, for any $K \in \mathfrak{T}$, the relation

$$B_{\sigma_0\, diam\, K}(x) \subset K \tag{6}$$

is valid for some $x \in K$. All estimates appearing in the following involve constants that do not depend on the grid except via the parameter σ_0 in (6). We put $h := \max\{\operatorname{diam} K \,:\, K \in \mathfrak{T}\}$. As a consequence of (6), we have

$$(\operatorname{diam} K)^2 \leq c|K| \quad \text{for } \; K \in \mathfrak{T}, \tag{7}$$

with $c > 0$ only depending on σ_0. Let $\mathfrak{S}$ be the set of the sides of the triangles $K \in \mathfrak{T}$. Put $J := \{1, ..., \#\mathfrak{S}\}$, where $\#\mathfrak{S}$ denotes the number of elements of $\mathfrak{S}$. Let $(S_i)_{i \in J}$ be a numbering of $\mathfrak{S}$, and denote the midpoint of S_i by Q_i $(i \in J)$. Set $J^o := \{i \in J \,:\, Q_i \in \Omega\}$, so that $J \backslash J^o = \{i \in J \,:\, Q_i \in \partial\Omega\}$. Note that for $i \in J \backslash J^o$, we have $S_i \subset \partial\Omega$.

We further introduce a barycentric mesh $(D_i)_{i \in J}$ on the triangular grid $\mathfrak{T}$: If $i \in J^o$, there are two triangles in $\mathfrak{T}$, denoted by K_i^1, K_i^2, such that $\overline{K_i^1} \cap \overline{K_i^2} = S_i$. We join the barycenter of each of these triangles with the endpoints of S_i. In this way we obtain a closed quadrilateral containing S_i (Fig. 1). This quadrilateral is denoted by D_i. If $i \in J \backslash J^o$ (hence $Q_i \in \partial\Omega$), let D_i be the closed triangle whose sides are the segment S_i and the segments joining the endpoints of S_i with the barycenter of the (unique) triangle $K \in \mathfrak{T}$ with $S_i \subset \overline{K}$. If $i, j \in J$ with $i \neq j$ are such that the set $D_i \cap D_j$ contains more than one point, then this set is a common side of D_i and D_j. In this case, the quadrilaterals D_i and D_j are called "adjacent", and their common side is denoted Γ_{ij}. For $i \in J$, we set

$$s(i) := \{j \in J \backslash \{i\} \,:\, D_i \text{ and } D_j \text{ are adjacent}\}.$$

If $i \in J$ and $j \in s(i)$, let n_{ij} denote the outward unit normal to D_i on Γ_{ij}. This means that n_{ij} points from D_i into D_j. We will use the abbreviations

$$\Theta_{ij}^+ := \int_{\Gamma_{ij}} \max\{\beta(x) \cdot n_{ij}, 0\}\, do_x \quad \overline{\mu}_i := \int_{D_i} \mu\, dx \quad (i \in J,\; j \in s(i)\,).$$

Since $n_{ij} = -n_{ji}$ $\big(i \in J,\; j \in s(i)\big)$ and because of $(3)_1$, we get

$$\sum_{j \in s(i)} (\Theta_{ij}^+ - \Theta_{ji}^+)/2 \,+\, \overline{\mu}_i = \sum_{j \in s(i)} \int_{\Gamma_{ij}} \beta(x) \cdot n_{ij}\, do_x/2 \,+\, \overline{\mu}_i \geq \int_{D_i} \nabla \cdot \beta\, dx/2 \,+\, \overline{\mu}_i \geq 0 \tag{8}$$

for $i \in J^o$. We introduce two finite element spaces by setting

$$X_h := \{v \in L^2(\Omega) \,:\, v|K \in P_1(K) \text{ for } K \in \mathfrak{T}_h, \;\; v \text{ continuous at } Q_i \text{ for } i \in J\},$$
$$V_h := \{v_h \in X_h \,:\, v_h(Q_i) = 0 \;\text{ for }\; i \in J \backslash J^o\},$$

where $P_1(A)$, for $A \subset \mathbb{R}^2$, denotes the set of all polynomials of degree at most 1 over A. The spaces X_h and V_h are nonconforming finite element spaces based on the piecewise linear Crouzeix-Raviart finite element.

From [7, (3.29), (3.31), (3.33)], we take a formula for the L^2-scalar product (v_h, w_h) of $v_h, w_h \in X_h$, that is,

$$(v_h, w_h) = \sum_{i \in J} v_h(Q_i)\, w_h(Q_i)\, |D_i|. \tag{9}$$

Put $H^1(\Omega) \oplus X_h := \{v + w_h \ : \ v \in H^1(\Omega),\ w_h \in X_h\}$, and let $I_h : H^1(\Omega) \oplus X_h \mapsto X_h$ be the interpolation operator introduced in [8, 8.9.79]; it is defined by

$$I_h(v) := \sum_{i \in J} l_i^{-1} \int_{S_i} v(x)\, do_x\, w_i \quad \text{for } \ v \in H^1(\Omega) \oplus X_h,$$

where l_i denotes the length of S_i $(i \in J)$. Note that a function $v \in H^1(\Omega)$ admits a trace on S_i for $i \in J$, and a function $v_h \in X_h$ verifies the equation

$$\int_{S_i} E(v_h|K_i^1)\, do_x = l_i\, v_h(Q_i) = \int_{S_i} E(v_h|K_i^2)\, do_x \quad (i \in J^o),$$

where $E(v_h|K_i^s)$ denotes the continuous extension of $v_h|K_i^s$ to S_i $(s \in \{1, 2\})$. Thus the operator I_h is well defined. By [8, Lemma 8.9.81], it satisfies the estimate

$$\|I_h(v)\|_2 \le \mathfrak{C} \|v\|_{1,2} \quad \text{for } \ v \in H^1(\Omega).$$

It will be useful to introduce another interpolation operator besides I_h. In fact, for $v \in L^2(\Omega)$ with $v|K \in C^0(K)$ for $K \in \mathfrak{T}_h$, v continuous at Q_i for $i \in J$, we set $\varrho_h(v) := \sum_{i \in J} v(Q_i)\, w_i$. Next we define a discrete convection term b_h, which is to approximate the variational form

$$b(v, w) := \int_\Omega \big(\nabla \cdot (v\,\beta) + \mu\, v \big)\, w\, dx \quad \big(v, w \in H^1(\Omega) \big)$$

associated with the convection term $\beta \cdot \nabla u$ in (1) and (4). We put

$$b_h(v_h, w_h) := \sum_{i \in J} w_h(Q_i) \Big(\sum_{j \in s(i)} \big(\Theta_{ij}^+ v_h(Q_i) - \Theta_{ji}^+ v_h(Q_j) \big) + \overline{\mu}_i\, v_h(Q_i) \Big) \quad \text{for } \ v_h, w_h \in X_h.$$

This definition means that we discretize b by an upwind finite volume method on the barycentric grid $(D_i)_{i \in J}$.

In view of discretizing the time variable, we fix $N \in \mathbb{N}$ and choose $t_1, \ldots, t_N \in (0, T)$ with $t_1 < \ldots < t_N$. Put $t_0 := 0$, $t_{N+1} := T$, $\tau_k := t_k - t_{k-1}$ for $1 \le k \le N+1$. For brevity, we put $G_h := I_h(G)$ and introduce functions $g_h^{(k)} : \overline{\Omega} \mapsto \mathbb{R}$ by setting

$$g_h^{(k)}(x) := I_h\big(g(\,\cdot\,, t_k) \big)(x) \quad \text{for } \ k \in \{0, \ldots, N+1\},\ x \in \overline{\Omega}.$$

Now we are in a position to introduce the finite element − finite volume discretization of problem (1), (2) and (4), (5), respectively, that we want to study in the work at hand. Concerning (1), (2), we consider functions $u_h^{(0)}, \ldots, u_h^{(N+1)} \in V_h$ with

$$\begin{aligned} &\tau_k^{-1} (u_h^{(k+1)} - u_h^{(k)}, v_h) + \nu\,((u_h^{(k+1)}, v_h))_{X_h} + b_h(u_h^{(k+1)}, v_h) = (g_h^{(k+1)}, v_h) \\ &\text{for } \ v_h \in V_h,\ k \in \{0, \ldots, N\}, \qquad u_h^{(0)} = I_h(u^{(0)}). \end{aligned} \tag{10}$$

This scheme is implicit because both the diffusion and the convection term are discretized implicitly. For the steady problem (4), (5), we consider an approximate solution $U_h \in V_h$ satisfying

$$\nu\,((U_h, v_h))_{X_h} + b_h(U_h, v_h) = (G_h, v_h) \quad \text{for } \ v_h \in V_h. \tag{11}$$

Our main results may be stated as follows.

Theorem 2.1 *Problem (10) admits a unique solution.*

Put $h_0 := \big(\underline{\beta}/(2C\varphi_1 \|\nabla\cdot\beta\|_p)\big)^{p/(p-2)}$, *where* $C>0$ *is some constant only depending on* σ_0. *Let* $u_h^{(0)}$, ..., $u_h^{(N+1)} \in V_h$ *be a system of functions satisfying (10). Then, if* $h \le h_0$,

$$\Big(\sum_{l=1}^{N+1} \tau_l \|u_h^{(l)}\|_2^2\Big)^{1/2} + \max_{1\le l\le N+1} \|u_h^{(l)}\|_2 + \nu^{1/2}\Big(\sum_{l=1}^{N+1} \tau_l \|u_h^{(l)}\|_{X_h}^2\Big)^{1/2} \tag{12}$$
$$\le \mathfrak{C}(1+\|\beta\|_{1,2})\Big[\Big(\sum_{l=1}^{N+1} \tau_l \|g_h^{(l)}\|_2^2\Big)^{1/2} + \|u_h^{(0)}\|_2\Big].$$

Theorem 2.2 *Problem (11) admits a unique solution. Let* $U_h \in V_h$ *be such a solution. Then, if* $h \le h_0$,

$$\|U_h\|_2 + \nu^{1/2}\|U_h\|_{X_h} \le \mathfrak{C}(1+\|\beta\|_{1,2})\|G_h\|_2,$$

with h_0 *from Theorem 2.1.*

3 A sketch of a proof of Theorem 2.1 and 2.2.

For $v_h \in V_h$, we put

$$\mathfrak{K}_h := \mathfrak{K}_h(v_h) := \sum_{i\in J}\sum_{j\in s(i)} \Theta_{ji}^+\big(v_h(Q_i) - v_h(Q_j)\big)^2,$$
$$\mathfrak{A}_h := \mathfrak{A}_h(v_h) := \sum_{j\in J} v_h(Q_i)^2 \sum_{j\in s(i)} \big(\Theta_{ji}^+\varphi(Q_i) - \Theta_{ij}^+\varphi(Q_j)\big),$$
$$\mathfrak{B}_h := \mathfrak{B}_h(v_h) := -\sum_{i\in J} v_h(Q_i)^2 \sum_{j\in s(i)} \int_{\Gamma_{ij}} \varphi(x)\,\beta(x)\cdot n_{ij}\,do_x.$$

Then

$$b_h(v_h,v_h) \ge \mathfrak{K}_h/2, \quad \text{in particular} \quad b_h(v_h,v_h)\ge 0 \quad \text{for } v_h\in V_h; \tag{13}$$

compare the proof of [3, Lemma 3.3];

$$\mathfrak{A}_h/2 + \sum_{i\in J} v_h(Q_i)^2\varphi(Q_i)\Big(\sum_{j\in s(i)} (\theta_{ij}^+ - \theta_{ji}^+) \ + \ \overline{\mu}_i\Big) \le b_h\big(v_h,\, \varrho_h(v_h\varphi)\big); \tag{14}$$

compare the proof of [3, Lemma 3.4];

$$\mathfrak{B}_h \ge \underline{\beta}\,\|v_h\|_2^2 - \sum_{i\in J} v_h(Q_i)^2 \int_{D_i} (\nabla\cdot\beta)(x)\,\varphi(x)\,dx; \tag{15}$$

compare the proof of [3, Lemma 3.5];

$$|\mathfrak{A}_h - \mathfrak{B}_h| \le \mathfrak{C}\|\beta\|_{1,2}^{1/2}\|v_h\|_2\,\mathfrak{K}_h^{1/2}; \tag{16}$$

see the proof of [3, Lemma 3.6]. The fact that $b_h(v_h,v_h)\ge 0$ for $v_h\in V_h$ (see (13)) implies the existence result in Theorem 2.1 and 2.2. From (13) – (16), we may conclude that

$$\underline{\beta}\,\|v_h\|_2^2 \le \mathfrak{C}\|\beta\|_{1,2}^{1/2}\|v_h\|_2\, b_h(v_h,v_h)^{1/2} + \sum_{i\in J} v_h(Q_i)^2\int_{D_i}(\nabla\cdot\beta)(x)\,\varphi(x)\,dx \tag{17}$$
$$+2b_h\big(v_h,\,\varrho_h(v_h\varphi)\big) - 2\sum_{i\in J} v_h(Q_i)^2\varphi(Q_i)\Big(\sum_{j\in s(i)}(\theta_{ij}^+ - \theta_{ji}^+) \ + \ \overline{\mu}_i\Big).$$

By (8), the two sums on the right-hand side of (17) are bounded by

$$\sum_{i\in J} v_h(Q_i)^2 \int_{D_i} (\nabla\cdot\beta)(x)\big(-\varphi(Q_i)+\varphi(x)\big)\,dx,$$

hence by

$$\|\nabla\varphi\|_\infty \sum_{i\in J} v_h(Q_i)^2\, diam\, D_i \int_{D_i} |(\nabla\cdot\beta)(x)|\, dx \le \varphi_1\, \|\nabla\cdot\beta\|_p \sum_{i\in J} v_h(Q_i)^2\, diam\, D_i\, |D_i|^{1/p'}.$$

For $i \in J$, let K_i^1 and K_i^2 be the two triangles K in $\mathfrak{T}$ with $Q_i \in K$. Then

$$\begin{aligned}
&\varphi_1\, \|\nabla\cdot\beta\|_p \sum_{i\in J} v_h(Q_i)^2\, diam\, D_i\, |D_i|^{1/p'} \le \varphi_1\, \|\nabla\cdot\beta\|_p \sum_{i\in J} v_h(Q_i)^2\, diam\, K_i^1\, |D_i|^{1/p'} \\
&\le C\, \varphi_1\, \|\nabla\cdot\beta\|_p\, h^{2/p'-1} \sum_{i\in J} v_h(Q_i)^2\, |D_i|,
\end{aligned}$$

In the last inequality we used (7), in the sense that $(diam\, K_i^1)^2 \le C\,|K_i^1| = C\,|D_i \cap K_i^1|/3 \le C\,|D_i|$. Here we write C for constants only depending on σ_0. It follows with (9) that

$$\varphi_1\, \|\nabla\cdot\beta\|_p \sum_{i\in J} v_h(Q_i)^2\, diam\, D_i\, |D_i|^{1/p'} \le C\, \varphi_1\, \|\nabla\cdot\beta\|_p\, h^{2/p'-1}\, \|v_h\|_2^2.$$

Returning to (17), we thus obtain

$$\underline{\beta}\, \|v_h\|_2^2 \le \mathfrak{C}\, \|\beta\|_{1,2}^{1/2}\, \|v_h\|_2\, b_h(v_h, v_h)^{1/2} + 2\, b_h\big(\, v_h,\ \varrho_h(v_h\, \varphi)\,\big) + C\, \varphi_1\, \|\nabla\cdot\beta\|_p\, h^{2/p'-1}\, \|v_h\|_2^2.$$

By the choice of h_0 in Theorem 2.1, we thus get in the case $h \le h_0$ that

$$\underline{\beta}\, \|v_h\|_2^2/2 \le \mathfrak{C}\, \|\beta\|_{1,2}^{1/2}\, \|v_h\|_2\, b_h(v_h, v_h)^{1/2} + 2\, b_h\big(\, v_h,\ \varrho_h(v_h\, \varphi)\,\big),$$

and then by another shoestring argument,

$$\|v_h\|_2^2 \le \mathfrak{C}\, (1 + \|\beta\|_{1,2})\big(\, b_h\big(\, v_h,\ \varrho(v_h\, \varphi)\,\big) + b_h(v_h, v_h)\,\big).$$

Now Theorem 2.1 and 2.2 follow with exactly the same arguments as used in [3] in order to prove [3, Theorem 2.1 and Theorem 2.2], respectively.

References

[1] Dolejší, V., Feistauer, M., Felcman, J. & Kliková, A.: Error estimates for barycentric finite volumes combined with nonconforming finite elements applied to nonlinear convection-diffusion problems. *Applications of Mathematics*, vol. 47: (2002) pp. 301–340.

[2] Feistauer, M., Felcman, J. & Straškraba, I.: Mathematical and computational methods for compressible flow. Clarendon Press, Oxford, 2003.

[3] Deuring, P., Eymard, R. & Mildner, M.: L^2-stability independent of diffusion for a finite element – finite volume discretization of a linear convection-diffusion equation. To appear in *SIAM Journal on Numerical Analysis*

[4] Barrenechea, G. R., John, V. & Knobloch, P.: A local projection stabilization finite element method with nonlinear crosswind diffusion for convection-diffusion-reaction equation. *Math. Model. Numer. Anal.*, vol. 47: (2013) pp. 1335–1366.

[5] Ayuso, B. & Marini, L. D.: Discontinuous Galerkin methods for advection-diffusion-reaction problems. *SIAM Journal on Numerical Analysis*, vol. 47: (2009) pp. 1391–1420.

[6] Devinatz, A., Ellis, R. & Friedman, A.: The asymptotic behavior of the first real eigenvalue of second order elliptic operators with a small parameter in the highest derivative. II. *Indiana University Mathematics Journal*, vol. 23: (1973) pp. 991–1011.

[7] Angot, P. , Dolejší, V., Feistauer, M. & Felcman, J.: Analysis of a combined barycentric finite volume – nonconforming finite element method for nonlinear convection-diffusion problems. *Applications of Mathematics*, vol. 43: (1998) pp. 263–310.

[8] Feistauer, M.: Mathematical methods in fluid dynamics. Pitman Monographs and Surveys in Pure and Applied Mathematics 67, Longman, Harlow, 1993.