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From the Finite Element Method
toward the Isogeometric Analysis
in an Object Oriented Computing Environment
Presentation Outline

- Motivation
- B-spline basis
- T-splines = NURBS + PB-splines
- Principles of OO design
- OOFEM
- OO design of IGA module
- Numerical example
- Summary
Motivation

Isogeometric Analysis

- recently introduced alternative to the FEM
- employs the same functions for the description of geometry and for the approximation of the solution on that geometry
  - eliminates costly FE mesh generation
  - geometric preprocessing still required
- outperforms classical FEM in various aspects
- still many open issues
  (trimmed geometry, boundary conditions, integration, efficiency issues, implementation, performance . . .)
Motivation

- IGA originally developed for NURBS
  - convenient for free-form surface modelling
  - exact representation of quadric surfaces
  - stable and efficient algorithms available
  - present in most CAD systems
  - gaps and overlaps cannot be avoided
  - trimmed NURBS not handled by IGA
  - generally only $C^0$ continuity across patch boundaries
  - tensor product structure of NURBS not efficient for representation of local features and for connection of adjacent surfaces
  - most shapes cannot be represented as a single watertight NURBS
Motivation

T-spline Based Isogeometric Analysis

- generalization of NURBS technology
  - inherits geometrical flexibility of NURBS
  - allows efficient local refinement
  - allows watertight merging of adjacent NURBS
  - T-splines are forward and backward compatible with NURBS
  - trimmed NURBS can be represented as T-spline
  - non-straightforward refinement around extraordinary points
  - non-trivial representation of solids preserving exactly boundary surface geometry
  - limited availability in commercial CAD (Maya, Rhino, SolidWorks)
Motivation

Implementation

- many similar features between FEM and IGA
- no need to start implementation from scratch
- most of the FE codes can be reused
- object oriented design recognized as very appropriate
  - proved to be a viable concept significantly enhancing modularity, extensibility, maintainability, and robustness of the code without sacrificing its performance
  - supports team work, allows further developments without participation of original authors
Univariate B-spline basis functions

\[ N_{i,p}(t) = \frac{t - t_i}{t_{i+p} - t_i} N_{i,p-1}(t) + \frac{t_{i+p+1} - t}{t_{i+p+1} - t_{i+1}} N_{i+1,p-1}(t) \quad \text{for } p > 0 \]

\[ N_{i,0}(t) = \begin{cases} 
1 & \text{if } t_i \leq t < t_{i+1} \\
0 & \text{otherwise} 
\end{cases} \]
Bivariate B-spline basis functions

\[ N_{ij,pq}^{uv}(u, v) = N_{i,p}^u(u)N_{j,q}^v(v) = N_k(u, v) \]

Rational bivariate B-spline basis functions

\[ R_k(t) = \frac{N_k(u, v)w_k}{\sum_{m=1}^{n} N_m(u, v)w_m} \quad k = 1, 2, \ldots, n \quad w_k > 0 \]
Quadratic NURBS curve

\[ r(t) = \sum_{j=1}^{6} R_i(t) P_i \]

\[ R_i(t) = \frac{N_i(t) w_i}{\sum_{j=1}^{6} N_j(t) w_j} \]

\[ t = \{0, 0, 0, 1, 3, 3, 4, 4, 4\} \]
NURBS – Nonuniform Rational B-splines

- a NURBS patch is defined by

- set of control points (coordinates and weights) topologically forming regular grid

- global degrees of B-spline basis functions for each parametric direction of the patch

- global knot vectors for each parametric direction of the patch
NURBS – Nonuniform Rational B-splines

- a NURBS patch is defined by
  - set of control points (coordinates and weights) topologically forming regular grid
  - global degrees of B-spline basis functions for each parametric direction of the patch
  - global knot vectors for each parametric direction of the patch

⇒ NURBS is fully structured
PB-splines – Point-based B-splines

- a PB-spline patch is defined by
  - set of control points (coordinates and weights) topologically irregular
  - local degrees of B-spline basis functions for each parametric direction of each control point
  - local knot vectors for each parametric direction of each control point
PB-splines – Point-based B-splines

- a PB-spline patch is defined by
  - set of control points (coordinates and weights)
    - topologically irregular
  - local degrees of B-spline basis functions for each parametric direction of each control point
  - local knot vectors for each parametric direction of each control point

⇒ PB-spline is fully unstructured
T-splines

- designed as compromise between NURBS and PB-splines
- a T-spline patch is defined by
  - set of control points (coordinates and weights) topologically consistent with a T-mesh
  - global degrees of B-spline basis functions for each parametric direction of the patch
  - global knot vectors for each parametric direction of the patch
T-splines

- designed as compromise between NURBS and PB-splines
- a T-spline patch is defined by
  - set of control points (coordinates and weights) topologically consistent with a T-mesh
  - global degrees of B-spline basis functions for each parametric direction of the patch
  - global knot vectors for each parametric direction of the patch

⇒ T-spline is quasi-structured
T-splines – local knot vector in parametric space
T-splines – local knot vector in parametric space
T-splines – local knot vector in parametric space
T-splines – local knot vector in index space

\[ u (3 – cubic) \]
\[ v (2 – quadratic) \]

\[ 0.0 \] \[ 0.2 \] \[ 0.4 \] \[ 0.8 \] \[ 1.0 \]
\[ 0.0 \] \[ 0.3 \] \[ 0.7 \] \[ 1.0 \]

\[ 0.0 \] \[ 0.2 \] \[ 0.4 \] \[ 0.8 \] \[ 1.0 \]
\[ 0.0 \] \[ 0.3 \] \[ 0.7 \] \[ 1.0 \]
T-splines – local knot vector in index space
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Object Oriented Design – Fundamental principles

- **encapsulation**
  (clustering together data and functionality)

- **inheritance**
  (reuse of existing code by derived classes)

- **abstraction / polymorphism**
  (transparent use of derived classes)

- **communication using messages**
  (general interface, safe data handling)

A good design is a trade-off between the level of implementation of object oriented principles and efficiency!
**OOFEM**

- **Object Oriented Finite Element Method** computing environment
- open source distributed under the GNU Public License
- being continuously developed since 1997
- inspired by FEM\_Object code (EPFL Lausanne, 1993)
- written in C++ ($\approx 185.000$ lines of code, $\approx 550$ classes)
- Ohloh analytics - 48 PersonYears

- modules for
  - structural mechanics
  - heat and mass transfer
  - fluid dynamics
OOFEM – Features

- fully extensible - a new element type, material model (with any internal history), BC, numerical algorithm, analysis module, ...
- independent problem formulation, numerical solution and data storage
- full restart support
- staggered analysis support
- parallel processing support - based on domain decomposition, message passing paradigms and dynamic load balancing
- adaptive analysis support
- eXtended FEM support
- efficient sparse solvers - interface to third party packages available
OO Design of IGA Module

- strict separation of
  - interpolation
  - integration
  - analysis-specific functionality

- implementation of general IGA element
- implementation of integration on IGA element
- implementation of interpolation on IGA element
- implementation of analysis-specific IGA element
OO Design of IGA Module

- FEInterpolation
- StructuralElementEvaluator
- PlaneStressStructuralElementEvaluator

- BSplineInterpolation
- NURBSInterpolation
- TSplineInterpolation

- int
- IntArray
- FloatArray

- dimension
- degree
- knotVector

- localIndexKnotVector
- numControlPoints

- iPointArray
- iRule
- iRuleArray

- knotSpan
- IGAIIntegrationElement

- GaussIntegrationRule

SIGA 2011
StructuralElementEvaluator::computeStiffnessMatrix(FloatMatrix answer) {
    element = this->giveElement();
    ndofs = element->giveNumberOfDofs();

    answer.resize(ndofs, ndofs);
    answer.zero();

    loop over all integration rules (iRule) on the element {
        loop over all Gauss points (gp) of the iRule {
            B = this->computeStrainDisplacementMatrix(gp);
            D = this->computeConstitutiveMatrix(gp);
            dV = this->computeVolumeAround(gp);
            answer->add(product of B^T.D.B.dV);
        }
    }
}
From the FEM toward the IGA in an OO Computing Environment
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**OO Design of IGA Module**

```cpp
PlaneStressStructuralElementEvaluator::
    computeStrainDisplacementMatrix(FloatMatrix answer, IntegPoint gp) {
        FEInterpolation interp = gp->giveElement()->giveInterpolation();
        interp->evalShapeFunctDerivatives(der, gp);
        nnodes = gp->giveElement()->giveNumberOfNodes();

        answer.resize(3, 2*nnodes); // 2 DOFs per each node
        answer.zero();

        for i=1:nnodes{
            answer.at(1, i*2-1) = der.at(i, 1); // dN(i)/dx
            answer.at(2, i*2) = der.at(i, 2); // dN(i)/dy
            answer.at(3, i*2-1) = der.at(i, 2); // dN(i)/dy
            answer.at(3, i*2) = der.at(i, 1); // dN(i)/dx
        }
    }
```
Numerical Example

\[ E = 15 \text{ GPa} \]
\[ \nu = 0.25 \]
\[ t = 0.15 \text{ m} \]
\[ \bar{u} = 1 \text{ m} \]
Numerical Example – IGA

Profile $\varepsilon_{xx}$
Numerical Example – IGA × FEA

IGA
3x2 T-spline
44 control points

IGA
5x5 NURBS
294 control points

FEA
bilinear quads
7345 nodes
7168 elements
Numerical Example – IGA × FEA – detail

IGA
3x2 T-spline
44 control points

IGA
5x5 NURBS
294 control points

FEA
bilinear quads
7345 nodes
7168 elements

profile $\varepsilon_{xx}$
Summary

- implementation of an IGA module into an existing object oriented finite element code was presented
- emphasis was given on proper OO design
  - most of the functionality of the existing code reused
  - modularity and extensibility of the code preserved
- amount of modified and/or added code is rather limited mostly related to handling basis functions
- functionality of implementation was verified on numerical example
- T-spline based IGA proved to be a promising technology
Acknowledgments

- implemented into open source FEM package OOFEM
  oofem.org

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