On the motion of rigid bodies in an incompressible or compressible viscous fluid under the action of gravitational forces

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1 Introduction and weak formulation

We consider the motion of several rigid bodies in a non-Newtonian fluid of power-law type (see Chapter 1 in Málek et al. \([M]\) for details), where the viscous stress tensor \(S\) depends on the symmetric part \(D[u], D[u] = \nabla_x u + \nabla^t_x u\) of the gradient of the velocity field \(u\) in the following way: Assumptions \((A1)\): \(S = S[D[u]], S : R^{3 \times 3}_{\text{sym}} \rightarrow R^{3 \times 3}\) is continuous, \((S[M] - S[N]) : (M - N) > 0\) for all \(M \neq N\), and \(c_1|M|^p \leq S[M]\); \(M \leq c_2(1 + |M|^p)\) for a certain \(p \geq 4\) and newtonian case for compressible case.

For the description of the initial position of the bodies see [DN]. The mass density \(\rho = \rho(t, x)\) and the velocity field \(u = u(t, x)\) at a time \(t \in (0, T)\) and the spatial position \(x \in \Omega\) satisfy the integral identity \(\int_0^T \int_\Omega \left( \rho \partial_t \phi + \rho u \cdot \nabla \phi \right) dx\, dt = -\int_\Omega \rho_0 \phi \, dx, \phi \in C^1([0, T] \times \Omega)\),
\[
\int_0^T \int_\Omega \left( \rho u \cdot \partial_t \phi + \rho u \otimes u : \nabla x \phi - S : D[\phi] \right) \, dx\, dt \\
= -\int_0^T \int_\Omega \rho G \nabla \phi \int_{R^3} \frac{\rho_j}{|x - y|^2} dy \cdot \phi \, dx\, dt - \int_\Omega \rho_0 u_0 \cdot \phi \, dx\, dt \\
\phi \in C^1([0, T] \times \Omega), \phi(t, \cdot) \in \mathcal{R}(t), \\
\mathcal{R}(t) = \{ \phi \in C^1(\Omega) \mid \text{div } \Phi = 0 \text{ in } \Omega, \phi = 0 \text{ on a neighborhood of } \partial \Omega, D[\Phi] = 0 \text{ on a neighborhood of } U_{\eta_i, \mathcal{B}_i(t)}, \text{ where } \int_0^T \int_\Omega \rho G \nabla x F \, dx\, dt = \int_0^T \int_\Omega \rho G \nabla x F \, dx\, dt, \text{ with } F = \left( \sum_{i \neq j} \int_{R^3} \frac{\rho_j}{|x - y|^2} dy \left( \int_{R^3} \frac{\rho_i^e}{|x - y|^2} dy \right) \right). \text{ Finally, we require the velocity field } u \text{ to be compatible with the motion of bodies. As the mappings } \eta_i(t, i) \text{ are isometries on } R^3, \text{ they can be written in the form } \eta_i(t, x) = x_i(t) + \mathcal{O}_i(t)x. \text{ Accordingly, we impose to the velocity field } u \text{ to be compatible with the family of motions } \{\eta_1, \ldots, \eta_n\} \text{ if } u(t, x) = u^{B_i}(t, x) = U_i(t) + \mathcal{O}_i(t)(x - x_i(t)) \text{ for a.a. } x \in \mathcal{B}_i(t), \text{ i = 1, \ldots, n for a.a. } t \in [0, T], \text{ where } \frac{d}{dt} x_i = U_i, \left( \frac{d}{dt} \mathcal{O}_i \right) \mathcal{O}^T_i = \mathcal{Q}, \text{ a.a. on } (0, T). \]

**Problem P**

Let the initial distribution of the density and the velocity field be determined through given \(\rho_0, u_0\), respectively. The initial position of the rigid bodies being \(B^i \subset \Omega, \text{ i = 1, \ldots, n}\). We say that a family \(\rho, u, \eta_i, i = 1, \ldots, n\), represent a variational solution of problem \((P)\) on a time interval \((0, T)\) if the following conditions are satisfied:
(1) The density $\rho$ is a non-negative bounded function, the velocity field $u$ belongs to the space $L^\infty(0,T;L^2(\Omega;R^3)) \cap L^p(0,T;W_0^{1,p}(\Omega;R^3))$, and they satisfy energy inequality (EI) for $t_1 = 0$ and a.e. $t_2 \in (0,T)$.

(2) The continuity equation holds on $(0,T) \times R^3$ provided $\rho$ and $u$ are extended to be zero outside $\Omega$.

(3) Momentum equation (the integral identity) holds for any admissible test function $w \in R(t)$.

(4) The mappings $\eta^i$, $i = 1,\ldots,m$ are affine isometries of $R^3$ compatible with the velocity field $u$ in the sense of compatibility conditions.

Let us formulate one of our main existence results.

**Theorem 1.1** Let the initial position of the rigid bodies be given through a family of open sets

$$B_i \subset \Omega \subset R^3, \ B_i \text{ diffeomorphic to the unit ball for } i = 1,\ldots,n,$$

where both $\partial B_i$, $i = 1,\ldots,n$, and $\partial \Omega$ belong to the regularity class see [DN]. In addition, suppose that

$$\text{dist}(B_i, B_j) > 0 \text{ for } i \neq j, \ \text{dist}(B_i, R^3 \setminus \Omega) > 0 \text{ for any } i = 1,\ldots,n$$

and we assume that boundary of $\Omega$ and $B_i$ belong to $C^{2,\nu}$, $\nu \in (0,1)$. Furthermore, let the viscous stress tensor $S$ satisfy hypotheses (A1), with $p \geq 4$.

Finally, let the initial distribution of the density be given as

$$\varrho_0 = \begin{cases} \varrho_f = \text{const} > 0 \text{ in } \Omega \setminus \cup_{i=1}^n B_i, \\ \varrho_B, \text{ on } S_i, \text{ where } \varrho_B \in L^\infty(\Omega), \ \text{ess inf}_B \varrho_B > 0, \ i = 1,\ldots,n, \end{cases}$$

while

$$u_0 \in L^2(\Omega;R^3), \ \text{div}_x u_0 = 0 \text{ in } D'(\Omega), \ D[u_0] = 0 \text{ in } D'(B_i;R^{3 \times 3}) \text{ for } i = 1,\ldots,n.$$

Then there exist a density function $\varrho$,

$$\varrho \in C([0,T];L^1(\Omega)), \ 0 < \text{ess inf}_\Omega \varrho(t,\cdot) \leq \text{ess sup}_\Omega \varrho(t,\cdot) < \infty \text{ for all } t \in [0,T],$$

a family of isometries $\eta_{i}(t,\cdot)\}_{i=1}^{n}$, $\eta_i(0,\cdot) = 1$, and a velocity field $u$,

$$u \in C_{\text{weak}}([0,T];L^2(\Omega;R^3)) \cap L^p(0,T;W_0^{1,p}(\Omega;R^3)),$$

compatible with $\{\eta_i\}_{i=1}^{n}$ in the sense specified in (3.7), (3.8), such that $\varrho$, $u$ satisfy the integral identity (3.3) for any test function $\varphi \in C^1([0,T) \times R^3)$, and the integral identity (3.4) for any $\varphi$ satisfying (3.5), (3.6).

For compressible case see [DN1].

**References**


[DN1] On the motion of rigid bodies in an incompressible or compressible viscous fluid under the action of gravitational forces, Preprint 2011.