Continuum Mechanics, part 4

Stress_1
- Engineering and Cauchy
- First and second Piola-Kirchhoff
- Example for 1D strain and stress
- Green-Naghdi stress rate
Stress measures, notation and terminology

The elementary force responsible for the deformation

In deriving a ‘proper’ stress measure we require its independence of rigid body motion.

The engineering stress does not possess this property.
STRESS TENSORS

In linear elasticity the stress is defined as a limiting ratio of an elementary force $dF$ and an elementary surface $dA$. It should be reminded that using this approach, the element force belongs to the deformed configuration $\tilde{T}$ while the force to the initial one, i.e. $T$.

In finite deformation theory we have to take into account the change of geometry and the stresses should be properly related to strain.
Let's define unit vectors \( \vec{n}_0 \) and \( \vec{n} \) perpendicular to elementary surfaces \( dA_0 \) and \( dA \) in initial and reference configurations \( ^{\circ}C \) and \( ^{t}C \) respectively.

The stress vectors are defined as limiting ratios

\[
T_0 = \frac{dP_0}{dA_0}, \quad T = \frac{dP}{dA}
\]

In linear continuum mechanics we take \( \frac{dP}{dA_0} \)

- **ENGINEERING STRESS**
From the elementary linear elasticity, we have already learned that stress vectors can be expressed by stress tensor components as follows (by Cauchy relation):

\[ \{T\} = [\sigma]^T \{m\} \]

\[
\begin{bmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_{21} & \sigma_{22} & \sigma_{23} \\
\sigma_{31} & \sigma_{32} & \sigma_{33}
\end{bmatrix}
= 
\begin{bmatrix}
\sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\
\sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\
\sigma_{zx} & \sigma_{zy} & \sigma_{zz}
\end{bmatrix}
= [\sigma]
\]
'vector' of these tensor components (computationally...away)

\[
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{zz} \\
\sigma_{xy} \\
\sigma_{yz} \\
\sigma_{zx}
\end{bmatrix} =
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_4 \\
\sigma_5 \\
\sigma_6
\end{bmatrix} = \{ \sigma \}
\]

Not vector in tensorial sense
The transformation law is not obeyed

Dimension Sigma(6)
Let $d\bar{F}$ is an elementary force responsible for the change of an elementary volume from $0^\circ C$ to $t^\circ C$. In the current configuration, we have
This choice will lead to the first Piola-Kirchhoff stress tensor.
Motivation for the definition of a fictive force in reference configuration

In the current configuration we know the applied forces, but the deformed geometry is unknown. The stress measure is clearly defined (Cauchy or true stress) but cannot be directly computed.

Inventing a fictive force ‘acting’ in the reference configuration (related in a systematic way to the actual force in the current configuration) allows to define a new suitable measure of stress in the reference configuration, (e.g. Piola-Kirchhoff) to compute it and relate it back to the the true stress. Such a stress should be independent of rigid body motion and of the choice of coordinate system.

The true stress is the only measure which is of final interest from engineering point of view.

Other measures are just useful tools to get the true stress.
the aching force can be related to original configuration as well (but it is a fiction)

\[
\{\mathbf{dP}_0\} = \{\mathbf{dP}\} = \{\mathbf{T}_0\} \, d\mathbf{A}_0 \quad \{\mathbf{T}_0\} = [\mathbf{C}]^T \{\mathbf{m}_0\}
\]

\[
\Rightarrow \quad \{\mathbf{dP}\} = [\mathbf{C}]^T \, d\mathbf{A}_0 \, \{\mathbf{m}_0\} \quad (**) \quad \text{first Piola-Kirchhoff or Lagrange stress tensor}
\]

Comparing (**) and (**) we get

\[
[\mathbf{G}]^T \, d\mathbf{A} \, \{\mathbf{m}_1\} = [\mathbf{C}]^T \, d\mathbf{A}_0 \, \{\mathbf{m}_0\} \quad (***)
\]

In order to find the relation between \([\mathbf{G}]\) and \([\mathbf{C}]\), it is necessary to find relation between \(d\mathbf{A}\) and \(d\mathbf{A}_0\).
Comparing volumes before \((dV_0)\) and after \((dV)\) the deformation and taking into account the law of conservation of mass we can write

\[ \int_0^1 dV_0 = \int_0^1 dV \]

The initial volume is

\[ dV_0 = \frac{1}{6} da_1 da_2 da_3 = \]

\[ = \frac{1}{9} \left( \frac{1}{2} da_2 da_3 da_1 + \frac{1}{2} da_1 da_3 da_2 + \frac{1}{2} da_1 da_2 da_3 \right) = \]

\[ \frac{1}{9} da_{01} \{ da \} \]

Notice \( \{ da \} = \{ da_0 \} \{ m_0 \} \)

\[ \overrightarrow{da}_0 = da_{0i} m_{0i} \]
\[ \text{So } dV_0 = \frac{1}{q} \{dA_0\}^T \{da\} \text{ and similarly } \\
\text{and } \\
dV = \frac{1}{q} \{dA\}^T \{dx\} \text{ initial and final volumes} \]

Now we can conclude

\[ \text{So } dV_0 = q dV \]

\[ \text{So } \{dA_0\}^T \{da\} = q \{dA\}^T \{dx\} \]

Since \{da\} is arbitrary

\[ \text{So } \{dA_0\}^T = q \{dA\}^T [F] \]

Using \& we get

\[ \text{So } dA_0 \{m_0\}^T = q \: dA \: \{m\}^T [F] \]

\[ dA_0 \{m_0\}^T = \frac{q}{p_0} \: dA \: \{m\}^T [F] \]
Substituting the left-hand side to (***) we can write

\[ [\mathbf{G}]^T \, d\mathbf{A} \{ \mathbf{m} \} = [\mathbf{C}]^T \, \frac{s}{s_0} \, d\mathbf{A} \, [\mathbf{F}]^T \{ \mathbf{m} \} \]

This equation must be valid for any \( d\mathbf{A} \{ \mathbf{m} \} \).

We can finally write (after transposition):

\[ [\mathbf{G}] = \frac{s}{s_0} \, [\mathbf{F}] \, [\mathbf{C}] \]

\[ [\mathbf{C}] = \frac{s_0}{s} \, [\mathbf{F}]^{-1} \, [\mathbf{G}] \]

\( J = \det [\mathbf{F}] = \frac{s_0}{s} \)

The first Piola-Kirchhoff is not symmetric
there is another possibility how to relate the stress tensor to original configuration, namely

\[ \{dP_0\} = [F]^{-1}\{dP\} \quad \text{as} \quad \{da\} = [F]^{-1}\{dx\} \]

A new measure is introduced here, the second Piola-Kirchhoff

\[ \{dP_0\} = [S]^T \text{d}A_0 \{m_0\} \]

\[ \{dP\} = [s]^T \text{d}A \{m\} \]

As before we get the second Piola-Kirchhoff defined by

\[ [S] = \frac{p_0}{p} [F]^{-1} [s] [F]^{-T} \]

\[ [S] = \frac{p_0}{p} [F] [s] [F]^T \]

\[ J = \det [F] = \frac{p_0}{p} \]
The second Kirchhoff-Piola stress tensor is a more suitable measure of state of stress than the first K-P. It is symmetric.

It should be noted that the second Piola-Kirchhoff stress tensor has little physical meaning. It is just a suitable measure which is energetically conjugate with Green-Lagrange main tensor. Which means that their product corresponds to work or energy.
Example for 1D strain and stress

GREEN-LAGRANGE STRAIN TENSOR FOR 1D PROBLEM

ASSUMED STRAIN DISTRIBUTION

Relation between coordinate systems \( t_x = t_x (0_x, t) \)
In this case:

\[ t_x = \frac{\Delta l}{l_0} \] \[ o_x = \frac{\Delta l + \Delta l}{l} \] \[ o_x = \left(1 + \frac{\Delta l}{l_0}\right) o_x \]

\[ t_x = \left(1 + \varepsilon_x\right) o_x \] \[ \varepsilon_x = \frac{\Delta l}{l_0} \]

Let's assume:

\[ u_x = \text{const} \ t_x \]

The unknown constant can be found from:

\[ \Delta l = \text{const} \ t_l \]

\[ \text{const} = \frac{\Delta l}{t l} \]

So the distribution of displacement is:

\[ u_x = \frac{\Delta l}{t l} t_x = \frac{\Delta l}{t l} \frac{t l}{\Delta l} o_x = \frac{\Delta l}{l_0} o_x = \varepsilon_x o_x \]
AND WHAT ABOUT CROSS-SECTION QUANTITIES

\[ \theta = \theta_0 + \Delta \theta \]

GEOMETRY:

\[ \frac{tA}{oA} = \left( \frac{tr}{or} \right)^2 = \left( 1 + \frac{\Delta r}{or} \right)^2 = \left( 1 + \varepsilon_r \right)^2 \]

RELATIONS BETWEEN COORDINATE SYSTEMS

\[ t_y = \text{const} \cdot o_y \]
\[ t_z = \text{const} \cdot o_z \]

\[ t_r = \text{const} \cdot o_r \]
\[ \text{const} = \frac{tr}{or} = \left( \frac{tA}{oA} \right)^{1/2} = \left( 1 + \varepsilon_r \right) \]
So:

\[ r_r = (1 + \varepsilon_r) \cdot r \]
\[ y_t = (1 + \varepsilon_r) \cdot y \]
\[ z_2 = (1 + \varepsilon_r) \cdot z \]

\[ y_t - y_0 = \varepsilon_r \cdot y \]
\[ y_2 - y_0 = \varepsilon_r \cdot z \]

**Displacement Distribution**
SO THE KINEMATIC RELATION IN THIS CASE IS

\[ t_{x_1} = t_x (0_{x_1} t) \]

\[ t_x = (1 + \varepsilon_x) \circ x \]
\[ t_y = (1 + \varepsilon_r) \circ y \]
\[ t_z = (1 + \varepsilon_r) \circ z \]

\[ t_{x_1} = (1 + \varepsilon_x) \circ x_1 \]
\[ t_{x_2} = (1 + \varepsilon_r) \circ x_2 \]
\[ t_{x_3} = (1 + \varepsilon_r) \circ x_3 \]
AND THE DEFORMATION GRADIENT IS

\[ F_{ij} = \frac{\partial x_i}{\partial x^j} = \begin{bmatrix} 1+\varepsilon_x & 0 & 0 \\ 0 & 1+\varepsilon_y & 0 \\ 0 & 0 & 1+\varepsilon_z \end{bmatrix} \]

AND ITS DETERMINANT

\[ J = \det [F] = (1+\varepsilon_x)(1+\varepsilon_y)^2 = \left(1 + \frac{\partial l}{\partial A}\right) \frac{\partial A}{\partial A} = \]

\[ \frac{\partial l + \Delta l}{\partial l} \frac{\partial A}{\partial A} = \frac{tl}{\partial l} + \frac{tA}{\partial A} \]
GREEN–LAGRANGE STRAIN TENSOR

\[ [E] = \frac{1}{2} \left( [F]^T [F] - I \right) = \]

\[
\begin{bmatrix}
1 + \varepsilon_x + \varepsilon_x + \varepsilon_x^2 - 1 \\
1 + \varepsilon_r + \varepsilon_r + \varepsilon_r^2 - 1 \\
\varepsilon_x + \frac{1}{2} \varepsilon_x \\
\varepsilon_r + \frac{1}{2} \varepsilon_r \\
\varepsilon_r + \frac{1}{2} \varepsilon_r
\end{bmatrix}
\]
IN THE TEXT THAT FOLLOWS WE WILL USE

\[ \varepsilon_{11} = \varepsilon_{GG} = \varepsilon_x + \frac{1}{2} \varepsilon_x^2 = \frac{tl^2 - 0l^2}{20l^2} = \frac{1}{2} \left( \varepsilon^2 - 1 \right) \]

ENGINEERING STRAIN

\[ \varepsilon_x = \frac{\Delta l}{0l} \]

STRETCH

\[ \varepsilon = \frac{tl}{0l} \]
DIFFERENT STRAIN MEASURES
FOR A 'THIN' BAR

\[ \varepsilon = \frac{\Delta l}{l} \]

\[ E_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right) = \varepsilon_{ij} \]

\[ A_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right) = \varepsilon_{ij} \]
11 components only!

For 1D continuum

\[
\begin{align*}
e_E &= \frac{1}{2} \frac{t \ell^2 - 0 \ell^2}{0 \ell^2} \\
e_A &= \frac{1}{2} \frac{t \ell^2 - 0 \ell^2}{t \ell^2 + 0 \ell - 0 \ell} \\
e_E &= \frac{1}{0 \ell} t \ell \\
e_L &= \log \left( \frac{t \ell}{0 \ell} \right)
\end{align*}
\]
The same physical phenomenon – different strain measures?
THE TRUE (CAUCHY) STRESS

\[ [\sigma] = \begin{bmatrix}
\frac{t_p}{t_A} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \]

THE SECOND PIOLA KIRCHHOFF STRESS

\[ [\mathbf{S}] = J [\mathbf{F}]^{-1} [\sigma] [\mathbf{F}]^{-T} = \]

\[ = \frac{t_l}{t_A} \frac{t_A}{t_A} \begin{bmatrix}
\frac{t_l}{t_A} & 1/(1+\varepsilon) \\
1/(1+\varepsilon) & 1/(1+\varepsilon)
\end{bmatrix} \begin{bmatrix}
\frac{t_p}{t_A} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \]

\[ = \frac{p_t}{t_A} \frac{t_A}{t_A} \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \]
PRINCIPLE OF VIRTUAL WORK

Green-Lagrange strain

\[ \delta E_e = \frac{t_e^2 - o_e^2}{2o_l^2} \]

\[ \int \delta E_e \, d\Omega = P_e \delta l \]

\[ P_e = 6e \frac{\delta l}{o_l} \]

\[ \delta E_e \]

Engineering strain

\[ P_e = 6e \, o_A \]

Actual force responsible for the change of configuration
Logarithmic strain

\[ \int_{V} \sigma_{L} \delta \varepsilon_{L} \, dV = P_{L} \delta \varepsilon_{L} \]

\[ P_{L} = 6_{L} \times A \]

\[ \Rightarrow 6_{L} = 6 \ldots \text{ true stress } = \frac{\varepsilon P}{\delta A} \]
but \( e_A = e_A \left( \frac{0.2}{0.01} \right)^2 \) and if \( v = 0.5 \), then \( P_L = 6L \cdot e_A \cdot 0.2 / \cdot 0.1 \).

All the forces are identical (it is the same physical phenomenon), so

\[ bP = P_E = P_C = P_L \]

and from it follows that
\[ \delta = \frac{\theta A}{t A} \theta l \delta_c \]

which is an equivalent of

\[ [\delta] = \frac{1}{J} [F][S][F]^T \]

shown before.
In discussing the stress and stress rate at large deformation we will examine three different measures.

1) the symmetric second Piola-Kirchhoff stress tensor \([S]\)

2) the "true" Cauchy stress \([\sigma]\)

3) and unrotated Cauchy stress \([\sigma_0]\)

The "true" Cauchy stress \([\sigma]\) is related to the second P.-K. stress \([S]\) by the relation which was already shown.
\[
[\sigma] = \frac{1}{J} [F] [S] [F]^T \quad \text{where} \quad J = \det [F] = \frac{p_0}{p}
\]

The unrotated Cauchy stress \([\tilde{\sigma}]\) can be expressed by

\[
[\tilde{\sigma}] = [R]^T [\sigma] [R] \quad \text{where} \quad [R] \text{ comes from } [F] \rightarrow [R][U]
\]

One should note that unrotated Cauchy stress \([\tilde{\sigma}]\) is the true stress associated with the biaxial \([U]\) alone. Since \([R]\) is a proper orthogonal tensor, the principal invariants of \([\sigma]\) and \([\tilde{\sigma}]\) are identical.
Understanding corotational stress

Old stress in global system $\sigma_{ij}^g$

Rotated stress in global system

$\sigma_{ij}^g = R_{ik} \sigma_{km}^g R_{jm}$

$\alpha = 30^\circ$

Unrotated (corotational) stress in a local system rotating with the body

$\sigma_{ij}^l = R_{ki} \sigma_{km}^g R_{mj}$
Green-Naghdi stress rate tensor

\[ \tau = R^T \sigma R \]

\( R \) is from \( F \rightarrow RU \) (polar decomposition)

\[ R^T R = I \]

\[ R^T = R^{-1} \]

rates:

\[ \dot{\tau} = \dot{R}^T \sigma R + R^T \dot{\sigma} R + R^T \sigma \dot{R} \]

It holds:

\[ \Omega = \dot{R} R^T \implies \dot{R} = \Omega R \]

- rate of rotation tensor (rate of rigid body rotation of a material particle)

\[ \dot{R}^T = R^T \Omega^T \]
From it follows

\[ \dot{t} = R^T \Omega^T \sigma R + R^T \dot{\sigma} R + R^T \sigma \Omega R \quad / R^T \text{ from right} \]

\[ \dot{t} R^T = R^T \Omega^T \sigma + R^T \dot{\sigma} + R^T \sigma \Omega \quad / R \text{ from left} \]

\[ R \dot{t} R^T = \Omega^T \sigma + \dot{\sigma} + \sigma \Omega \]

\[ \Omega \text{ je antisymmetric } \Rightarrow \Omega^T = -\Omega \]

\[ \begin{bmatrix} \dot{\sigma} \\ \dot{\Omega}^{GN} \end{bmatrix} = R \dot{t} R^T = \dot{\sigma} - \Omega \sigma + \sigma \Omega \]

This is Green-Naghdi. If \( \Omega \) is used instead of \( W \) then we have so called Jaumann stress rate spin tensor \( W \) is antisymmetric part of \( L \) (velocity gradient) and represents rate of rotation of principal axis of \( D \) (rate of deformation).

\[ \begin{bmatrix} \dot{\sigma} \\ \dot{\sigma} \end{bmatrix}^{Ja} = \dot{\sigma} - W \sigma + \sigma W \]

\[ \mathcal{W} = \frac{1}{2} (L - L^T) \]

\[ \frac{\partial \mathcal{W}}{\partial x_i} = \frac{\partial}{\partial x_i} \frac{\partial \mathcal{W}}{\partial x_j} \]